

# Scattering of Glue by Glue on the Light-cone Worldsheet II: Helicity Conserving Amplitudes<sup>1</sup>

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## Abstract

This is the second of a pair of articles on scattering of glue by glue, in which we give the light-cone gauge calculation of the one-loop on-shell helicity conserving scattering amplitudes for gluon-gluon scattering (neglecting quark loops). The  $1/p^+$  factors in the gluon propagator are regulated by replacing  $p^+$  integrals with discretized sums omitting the  $p^+ = 0$  terms in each sum. We also employ a novel ultraviolet regulator that is convenient for the light-cone worldsheet description of planar Feynman diagrams. The helicity conserving scattering amplitudes are divergent in the infra-red. The infrared divergences in the elastic one-loop amplitude are shown to cancel, in their contribution to cross sections, against ones in the cross section for unseen bremsstrahlung gluons. We include here the explicit calculation of the latter, because it assumes an unfamiliar form due to the peculiar way discretization of  $p^+$  regulates infrared divergences. In resolving the infrared divergences we employ a covariant definition of jets, which allows a transparent demonstration of the Lorentz invariance of our final results. Because we use an explicit cutoff of the ultraviolet divergences in exactly 4 space-time dimensions, we must introduce explicit counterterms to achieve this final covariant result. These counter-terms are polynomials in the external momenta of the precise order dictated by power-counting. We discuss the modifications they entail for the light-cone worldsheet action that reproduces the “bare” planar diagrams of the gluonic sector of QCD. The simplest way to do this is to interpret the QCD string as moving in six space-time dimensions.

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# 1 Introduction

This is the second of a pair of articles on scattering of glue by glue to one loop in the language of the lightcone worldsheet [1–3]. The first article (I) [4] worked out the (finite) one loop amplitudes for helicity nonconserving processes. In this article we extend the calculation to the helicity conserving case, for which it is necessary to deal with ultraviolet and infrared divergences.

We refer the reader to the introduction of I for the detailed motivation and background for this work. Here we briefly mention the highlights. The goal of the program to give a worldsheet description of the sum of planar diagrams is to shed light on Field/String duality from the “field” side at weak coupling. This is just the perturbation expansion of the field theory. The mapping of the sum of planar diagrams to a worldsheet system [2, 3] in essence allows one to read off the worldsheet dynamics for scalar field theory and Yang-Mills theory in this weak coupling limit. A serious limitation of these initial articles, however, is that they transcribed the “bare” Feynman diagrams, without including any of the counterterms necessary to maintain gauge invariance. In the context of dimensional regularization this limitation is innocuous, because dimensional regularization automatically includes them. However, we want the worldsheet formalism to work in four dimensions, and so we must have in the worldsheet action the flexibility to include counterterms that go beyond the initial input Lagrangian. In [5] the ultraviolet structure of  $\phi^3$  theory was analyzed on the lightcone and it was shown, to all orders in perturbation theory, that two new counterterms in addition to those associated with mass, wave function, and coupling renormalization were necessary and sufficient. Happily, a local modification of the “bare” worldsheet action allowed for these new terms.

The aim of I and the present article is to execute the same program for Yang-Mills theory in light-cone gauge. Because the corresponding analysis is considerably more complex, we have limited their scope to one loop. In I we focused on one-loop helicity-violating amplitudes for which the on-shell tree diagrams vanished. As a consequence the one loop amplitudes are finite in both the ultraviolet and infrared. We could therefore confirm that the worldsheet description produces the correct known answers without dealing with collinear and soft gluon emission processes.

In contrast, the helicity-conserving processes studied in the present article display the full infrared divergence structure of non-abelian gauge theory. Just as in I, our infrared regulator is discretization of the  $p^+$  integrals omitting the  $p^+ = 0$  terms. But for the one-loop amplitudes we deal with here, this is essentially equivalent to simply reserving the  $p^+$  integrations to last. This is because all of the *artificial*  $p^+ = 0$  divergences actually cancel algebraically if the integrands from all diagrams, with momentum routed appropriately, are combined before the loop integrations are performed [6]. However, the true infrared divergences, for which  $p^+$  discretization provides a temporary regulator, are not cancelled in this combination, but rather are organized to cancel against divergences due to the absorption and emission of real soft and collinear gluons according to the Lee-Nauenberg theorem [7]. In addition, of course, these amplitudes have ultraviolet divergences which are taken care of by the renormalization program [8, 9]. In this article we give a complete lightcone gauge calculation of the scattering of glue by glue, including the processes involving extra gluons that resolve the infrared divergences. We work in four dimensions using the worldsheet friendly ultra-violet cutoff employed in [4, 5]. We organize the Feynman diagrams of the  $SU(N_c)$  Yang-Mills theory according to ‘t Hooft’s large  $N_c$  expansion [1], and we calculate the one-loop planar diagrams surviving the  $N_c \rightarrow \infty$  limit. The ‘t Hooft limit suppresses diagrams with quark loops, so they are not included here.

We remark here that our non-traditional methods of regulating and dealing with divergences are not sheer perversities on our part, but rather they are guided by the desire to fit these calculations into the framework of the lightcone worldsheet formalism. We hope the reader will bear with us on this point, and we assure her that, although unconventional, we have been extremely careful with the well-known subtleties and pitfalls of this difficult subject [10, 11].

To keep this paper reasonably self-contained, we repeat two short sections from I that summarize the lightcone Feynman rules (Section 2) and some useful identities (Section 3). A brief Section 4 lists all the four gluon trees. Then in Section 5 we discuss bremsstrahlung processes. We use a covariant definition of jets that proves to be very helpful in achieving nice compact results for these processes. We deal with initial state collinear (mass) divergences as in the original Lee-Nauenberg paper [7], where they are cancelled by including extra near collinear gluons in the initial state. This is in contrast to the standard technique,

used in analysing jets experimentally, that absorbs the initial state collinear divergences into the initial state parton distribution function [11]. Section 6 briefly summarizes the results for triangle and swordfish diagrams obtained in [12]. Section 7 contains the meat of this paper, the calculation of box diagrams. The reduction procedure developed in [4] for the helicity violating box diagrams is helpful for this. Section 8 describes the results of calculating the remaining quartic triangle and double quartic diagrams, and Section 9 finally puts everything together, and explains the cancellation of infrared divergences in cross sections. Section 10 discusses the problem of giving a local worldsheet description of all the counterterms necessary for Lorentz invariance. We find this can be simply done by interpreting the “string” dynamics of the worldsheet as occurring in six dimensional spacetime. The two added dimensions are holographically generated on the “string” side of Field/String duality and play no role on the “field” side. A final Section 11 wraps up the paper with a brief look at issues still to resolve in the future.

The reader who does not wish to follow the technical details of this work may get a glimpse of the main results and their impact on the lightcone worldsheet dynamics by simply reading sections 2, 9, 10 and 11. We particularly would like to draw his attention to the nice and compact final results for the infrared finite and Lorentz covariant probabilities (more precisely, the unnormalized squared amplitudes for jet-jet scattering) Eqs. (121), (122) of Section 9.2. The implications of the necessary counterterms for the worldsheet dynamics, which is the subject of Section 10, should also be amusing for this reader.

## 2 Feynman Rules for Light-cone gauge Yang-Mills

Here, we use the notation and conventions in Ref. [13], according to which the values of the non-vanishing three transverse gluon vertices are:

$$\begin{array}{c} \text{Diagram: A vertex with three lines. Two lines enter from the bottom (labeled 1 and 2) and one line exits from the top. The lines are slightly curved to indicate helicity.} \end{array} = \frac{2gp_3^+}{p_1^+ p_2^+} (p_1^+ p_2^\wedge - p_2^+ p_1^\wedge) = \frac{2gp_3^+}{p_1^+ p_2^+} K_{12}^\wedge \quad (1)$$

$$\begin{array}{c} \text{Diagram: A vertex with three lines. Two lines enter from the bottom (labeled 1 and 2) and one line exits from the top. The lines are slightly curved to indicate helicity.} \end{array} = \frac{2gp_3^+}{p_1^+ p_2^+} (p_1^+ p_2^\vee - p_2^+ p_1^\vee) = \frac{2gp_3^+}{p_1^+ p_2^+} K_{12}^\vee \quad (2)$$

The quartic vertices in this helicity basis are given by

$$\begin{array}{c} \text{Diagram: A four-point vertex with four lines. Two lines enter from the bottom (labeled 1 and 2) and two lines exit from the top (labeled 3 and 4). The lines are slightly curved to indicate helicity.} \end{array} = -2g^2 \frac{p_1^+ p_3^+ + p_2^+ p_4^+}{(p_1^+ + p_4^+)^2} \quad (3)$$

$$\begin{array}{c} \text{Diagram: A four-point vertex with four lines. Two lines enter from the bottom (labeled 1 and 2) and two lines exit from the top (labeled 3 and 4). The lines are slightly curved to indicate helicity.} \end{array} = +2g^2 \left( \frac{p_1^+ p_2^+ + p_3^+ p_4^+}{(p_1^+ + p_4^+)^2} + \frac{p_1^+ p_4^+ + p_2^+ p_3^+}{(p_1^+ + p_2^+)^2} \right) \quad (4)$$

In these expressions,  $p_k^\wedge = (p_k^x + ip_k^y)/\sqrt{2}$ ,  $p_k^\vee = (p_k^x - ip_k^y)/\sqrt{2}$ , and  $p_k^+ = (p_k^0 + p_k^z)/\sqrt{2}$  are momenta *entering* the diagram on leg  $k$ , and  $g$  is proportional to the conventional QCD coupling  $g_s$ . Note that these are light-cone gauge ( $A_- = 0$ ) expressions and include the contributions that arise when the longitudinal field  $A_+$  is eliminated from the formalism.<sup>5</sup> We also should point out that we are giving these rules in the context of the 't Hooft's  $1/N_c$  expansion at fixed  $N_c g_s^2$ . Then the *planar* diagrams of the  $SU(N_c)$  theory are correctly given

<sup>5</sup>These vertex rules are convenient for the mixed  $\tau = ix^+, p^+, \mathbf{p}$  representation used in the imaginary  $x^+$  worldsheet formalism [2], in which an  $i$  from each vertex has been absorbed in each  $dx^+$ :  $idx^+ = d\tau$  and the propagator is  $(2p^+)^{-1} e^{-\tau \mathbf{p}^2/2p^+}$ . It will sometimes be convenient in this paper to return to full Minkowski momentum space  $p^-, p^+, \mathbf{p}$ . Then with the vertex rules given here and no  $i$ 's in the momentum space propagators  $1/(p^2 - i\epsilon)$  with  $p^2 = \mathbf{p}^2 - 2p^+ p^-$ , each Minkowski loop momentum integral should be accompanied by a  $-i$ :  $-id^4 q_M$ . With a further Wick rotation to Euclidean space  $d^4 q_M = id^4 q_E$  the  $i$ 's would disappear entirely.

if we take  $g \equiv g_s \sqrt{N_c/2}$ . Non-planar diagrams with this definition of  $g$  must be accompanied by appropriate powers of  $1/N_c^2$ , depending on the number of “handles” in the diagram. We have not spelled the details out here, because our focus will be on the planar diagrams in this article. The results we obtain should therefore be compared to the limit  $N_c \rightarrow \infty$ , fixed  $g_s^2 N_c$  of those in the literature. In making such comparisons, note that our definition of  $g$  multiplies conventionally defined  $n$ -gluon tree amplitudes by a factor  $N_c^{n/2-1} \rightarrow N_c$  for  $n = 4$ , so for each gluon scattering process we remove this factor before comparing to the literature.

### 3 K Identities

As we have seen, the quantities

$$K_{ij}^\mu \equiv p_i^+ p_j^\mu - p_j^+ p_i^\mu \quad (5)$$

play a central role in the cubic Yang-Mills vertex. In fact, we shall find that the simplest forms of the various helicity amplitudes are achieved by expressing them as functions of the  $K$ 's. These simple forms are in fact identical to those achieved by Parke and Taylor using a bispinor representation of polarization vectors as the now famous Parke-Taylor amplitudes [14]. For us the role of the spinor matrix elements in those formulae will be played exclusively by  $K_{ij}^\wedge$  and  $K_{ij}^\vee$ .

In order to reduce the expressions for the helicity amplitudes to the Parke-Taylor form, we will need a number of identities enjoyed by the  $K$ 's. For a general  $n$ -gluon amplitude we can form  $K_{ij}$  for each pair of gluons  $(ij)$ , where  $i, j = 1, \dots, n$  distinguish the different gluons. By momentum conservation, it is immediate that

$$\sum_j K_{ij}^\mu = 0. \quad (6)$$

From the fact that  $K$  is an anti-symmetric product we have Bianchi-like identities

$$p_i^+ K_{jk}^\mu + p_k^+ K_{ij}^\mu + p_j^+ K_{ki}^\mu = 0 \quad (7)$$

$$K_{li}^\wedge K_{jk}^\wedge + K_{lk}^\wedge K_{ij}^\wedge + K_{lj}^\wedge K_{ki}^\wedge = 0 \quad (8)$$

Finally, the most powerful type of identity follows from a very simple calculation

$$\sum_j \frac{K_{ij}^\wedge K_{jk}^\vee}{p_j^+} = -p_i^+ p_k^+ \sum_j \frac{p_j^2}{2p_j^+} \quad (9)$$

which seems like a complicated non-linear relation. However when we are considering scattering amplitudes, the momenta all satisfy  $p_i^2 = 0$  so the right side is zero! This identity plays a central role in showing that trees with all but one like helicities vanish. (Trees with all like helicity can't even be drawn.) They are also crucial for reducing the complexity of the helicity amplitudes that don't vanish.

### 4 Summary of Tree Amplitudes

In this section we simply list the four point trees amplitudes on and off shell obtained in [4]. There are no tree diagrams for  $\wedge \wedge \wedge \wedge$  or  $\vee \vee \vee \vee$  polarizations. The off-shell  $\wedge \wedge \wedge \vee$  four-point tree is given by, omitting the coupling factor  $2g$  for each vertex,

$$\begin{aligned} A_{\wedge \wedge \wedge \vee}^{tree} &= -\frac{p_4^+}{p_1^+ p_2^+ p_3^+} \left[ \frac{K_{32}^\wedge K_{14}^\wedge}{(p_2 + p_3)^2} + \frac{K_{43}^\wedge K_{21}^\wedge}{(p_1 + p_2)^2} \right] \\ &= -\frac{p_4^+ (K_{43}^\wedge K_{32}^\wedge p_1^2 + K_{14}^\wedge K_{43}^\wedge p_2^2 + K_{21}^\wedge K_{14}^\wedge p_3^2 + K_{32}^\wedge K_{21}^\wedge p_4^2)}{p_1^+ p_2^+ p_3^+ (p_1 + p_2)^2 (p_2 + p_3)^2} \rightarrow 0 \quad \text{On shell} \end{aligned} \quad (10)$$

The notation here is that  $\wedge$  denotes an incoming arrow representing helicity  $+1$ , while  $\vee$  denotes an outgoing arrow representing helicity  $-1$ .

The only non-zero four point trees are those with two of each helicity. There are two distinct helicity patterns. The amplitude for adjacent helicity  $\wedge\wedge\vee\vee$  is given by

$$A_{\wedge\wedge\vee\vee}^{tree} = -\frac{1}{(p_1^+ + p_4^+)^2} \left[ \frac{p_1^+ p_3^+}{p_2^+ p_4^+} \frac{K_{14}^\vee K_{32}^\wedge}{(p_1 + p_4)^2} + \frac{p_2^+ p_4^+}{p_1^+ p_3^+} \frac{K_{14}^\wedge K_{32}^\vee}{(p_1 + p_4)^2} + \frac{p_1^+ p_3^+ + p_2^+ p_4^+}{2} \right] \\ - \frac{(p_1^+ + p_2^+)^2 K_{21}^\wedge K_{43}^\vee}{p_1^+ p_2^+ p_3^+ p_4^+ (p_1 + p_2)^2} \quad (11)$$

When some legs are off-shell, we use the shorthand notation  $p_i^* \equiv p_i^2/p_i^+$  to simplify the writing.

$$A_{\wedge\wedge\vee\vee}^{tree} = -\frac{2K_{21}^{\wedge 2} K_{43}^{\vee 2}}{p_1^+ p_2^+ p_3^+ p_4^+ (p_1 + p_2)^2 (p_1 + p_4)^2} \\ + \frac{(p_1^+ + p_2^+)(p_1^* + p_2^* - p_3^* - p_4^*)}{2(p_1 + p_4)^2} - \frac{p_2^+ p_4^+ (p_1^* - p_3^*) + p_1^+ p_3^+ (p_4^* - p_2^*)}{2(p_1^+ + p_4^+)(p_1 + p_4)^2} \quad (12)$$

$$+ \frac{(p_1^+ + p_2^+)^2 (p_1^* + p_2^*)(p_3^* + p_4^*)}{2(p_1 + p_2)^2 (p_1 + p_4)^2} + \frac{K_{21}^\wedge K_{43}^\vee (p_1^+ + p_2^+)[p_1^+ p_3^+ (p_1^* - p_3^*) + p_2^+ p_4^+ (p_2^* - p_4^*)]}{p_1^+ p_2^+ p_3^+ p_4^+ (p_1 + p_2)^2 (p_1 + p_4)^2} \\ \rightarrow -\frac{2K_{21}^{\wedge 2} K_{43}^{\vee 2}}{p_1^+ p_2^+ p_3^+ p_4^+ (p_1 + p_2)^2 (p_1 + p_4)^2} \quad (\text{On Shell}) \quad (13)$$

The other distinct helicity arrangement for four gluon scattering is alternating helicity  $\wedge\vee\wedge\vee$ :

$$A_{\wedge\vee\wedge\vee}^{tree} = -\frac{1}{(p_1^+ + p_4^+)^2} \left[ \frac{p_1^+ p_2^+}{p_3^+ p_4^+} \frac{K_{14}^\vee K_{32}^\wedge}{(p_1 + p_4)^2} + \frac{p_3^+ p_4^+}{p_1^+ p_2^+} \frac{K_{14}^\wedge K_{32}^\vee}{(p_1 + p_4)^2} - \frac{p_1^+ p_2^+ + p_3^+ p_4^+}{2} \right] \\ - \frac{1}{(p_1^+ + p_2^+)^2} \left[ \frac{p_1^+ p_4^+}{p_2^+ p_3^+} \frac{K_{43}^\wedge K_{21}^\vee}{(p_1 + p_2)^2} + \frac{p_2^+ p_3^+}{p_1^+ p_4^+} \frac{K_{43}^\vee K_{21}^\wedge}{(p_1 + p_2)^2} - \frac{p_1^+ p_4^+ + p_2^+ p_3^+}{2} \right] \quad (14)$$

where the quartic vertex contribution has been split between the last terms in each of the square brackets. Notice that the second line on the right side is obtained from the first line with the relabeling substitutions  $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$  and  $\wedge \rightarrow \vee \rightarrow \wedge$ . Furthermore the first line can be obtained from the first line on the right of (11) by interchanging  $2 \leftrightarrow 3$  and multiplying by the factor  $-1$ . Thus by inspection we immediately obtain the simplifications

$$A_{\wedge\vee\wedge\vee}^{tree} = -\frac{2K_{31}^{\wedge 2} K_{42}^{\vee 2}}{p_1^+ p_2^+ p_3^+ p_4^+ p_{12}^2 p_{14}^2} - \frac{p_{13}^2 p_{24}^+ (p_1^* + p_3^*)(p_2^* + p_4^*)}{2p_{12}^2 p_{14}^2} + \frac{p_{13}^2 [p_4^+ p_2^* + p_2^+ p_4^* + p_3^+ p_1^* + p_1^+ p_3^*]}{2p_{12}^2 p_{14}^2} \\ - K_{31}^\wedge K_{42}^\vee \frac{p_3^+ p_4^+ (p_1^2 + p_2^2) + p_1^+ p_2^+ (p_3^2 + p_4^2) - p_2^+ p_3^+ (p_1^2 + p_4^2) - p_1^+ p_4^+ (p_2^2 + p_3^2)}{p_1^+ p_2^+ p_3^+ p_4^+ (p_1 + p_4)^2 (p_1 + p_2)^2} \\ + \frac{p_3^+ p_4^+ (p_1^* - p_2^*) + p_1^+ p_2^+ (p_4^* - p_3^*)}{2p_{14}^2 p_{14}^+} + \frac{p_1^+ p_4^+ (p_2^* - p_3^*) + p_2^+ p_3^+ (p_1^* - p_4^*)}{2p_{12}^2 p_{12}^+} \\ \rightarrow -\frac{2K_{31}^{\wedge 2} K_{42}^{\vee 2}}{p_1^+ p_2^+ p_3^+ p_4^+ p_{12}^2 p_{14}^2} \quad (\text{On Shell}) \quad (15)$$

Here and in the following we use the shorthand notation  $p_{i,i+1} = p_i + p_{i+1}$ , with  $i = 0, 1, 2, 3$  and  $p_4 \equiv p_0$ .

## 5 Gluon Bremsstrahlung and Jets

The consistent resolution of infrared divergences in loop corrections to scattering amplitudes involves a cancellation against corresponding infrared divergences in the cross section for the emission (or absorption)

of an extra gluon, whose momentum is either collinear with one of the gluons in the core process or “soft”. If the core process is scattering of glue by glue, the associated bremsstrahlung amplitudes are 5 gluon amplitudes.

In the context of the large  $N_c$  limit it is necessary to combine coherently only the bremsstrahlung diagrams with the same cyclic ordering. So, for example, in the diagrams shown in Fig. 1 at  $N_c = \infty$  it is only necessary to square the sum of the two diagrams on each line and combine the results on different lines incoherently. Because  $N_c = \infty$  suppresses nonplanar diagrams, a gluon line attached between two outgoing gluons (as

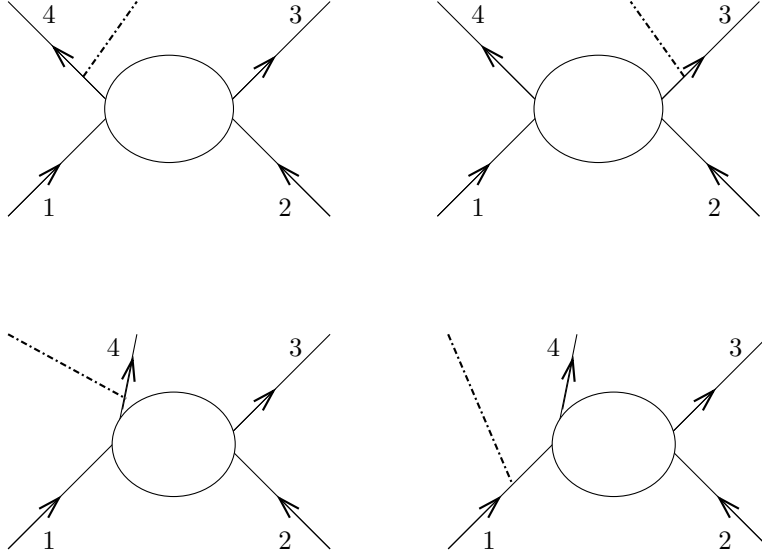


Figure 1: The bremsstrahlung diagrams associated with glue-gluon scattering involving leg 4. At  $N_c = \infty$  the sum of the diagrams on each line may be independently squared to give the leading contribution to the cross section. Similar pairs of diagrams involving each of the other legs must also be included.

with the diagrams on the first line of Fig. 1) must be outgoing when  $N_c = \infty$ <sup>6</sup>. Similarly a gluon line attached between two incoming gluons must be incoming. On the other hand, a gluon attached between an incoming and an outgoing gluon (as with the diagrams on the second line of Fig. 1) may be either incoming or outgoing.

As is well known infrared and collinear divergences are present only when the bremsstrahlung gluon attaches to external legs. For example if the brem gluon is collinear with  $p_4$ , there is a collinear divergence in the phase-space integral of the square of the diagrams where the gluon is emitted from or absorbed by leg 4. Calling the brem gluon’s four-momentum  $k$ , for fixed  $k^+$  the collinear point is  $\mathbf{k} = k^+ \mathbf{p}_4 / p_4^+$ . Then it is convenient to write

$$\mathbf{k} = k^+ \frac{\mathbf{p}_4}{p_4^+} + \hat{\mathbf{k}} \quad (16)$$

and examine the phase space integral for  $|\hat{\mathbf{k}}|$  in a neighborhood of zero. Here we assume  $k^+ = O(1)$  so the brem gluon is not soft. In effect, rather than measuring a single gluon, we insist that we measure a “jet” of total transverse momentum  $\mathbf{P} \approx (k^+ + p_4^+) \mathbf{p}_4 / p_4^+$  within a resolution  $\Delta$  [15]. A simple calculation shows

<sup>6</sup>At first glance the reader might think that the distinction between an incoming or outgoing line has nothing to do with planarity. But a well-defined large  $N_c$  limit only makes sense for physical quantities that are singlets under the gauge group. We specify our planar scattering amplitudes by imagining that the external gluons are all attached to a huge connected Wilson loop including a single connected portion at late times and another single connected portion at early times. The large  $N_c$  limit of such a quantity has the properties we describe in this paragraph.

that

$$\frac{(p_4^+ \vec{k} - k^+ \vec{p}_4)^2}{|k^+||p_4^+|} = 2|\vec{k}||\vec{p}_4|(1 - \cos \theta) \quad (17)$$

where the overarrow denotes the 3 spatial components of a four-vector and  $\cos \theta = \vec{k} \cdot \vec{p}_4 / |\vec{k}||\vec{p}_4|$ . The left side is thus a nice measure of the angular size of a jet, so we define the phase space of a jet of resolution  $\Delta$  by the restriction

$$\frac{(p_4^+ \vec{k} - k^+ \vec{p}_4)^2}{|k^+||p_4^+|} < \Delta^2 \quad (18)$$

This translates to  $\hat{\mathbf{k}}^2 < |k^+|\Delta^2/|p_4^+|$ .

The amplitudes for the emission of a hard brem gluon from the right of leg 4 (as in the first diagram on the first line of Fig. 1) are given, for the two polarizations, by

$$A_{\text{Brem}}^\vee = -2g \frac{k^+ + p_4^+}{k^+ p_4^+} \frac{K_{k,4}^\vee A_{\text{Core}}(p_1, p_2, p_3, k + p_4)}{(k + p_4)^2}, \quad \text{Outgoing helicity} \quad (19)$$

$$A_{\text{Brem}}^\wedge = -2g \frac{p_4^+}{k^+(k^+ + p_4^+)} \frac{K_{k,4}^\wedge A_{\text{Core}}(p_1, p_2, p_3, k + p_4)}{(k + p_4)^2}, \quad \text{Incoming helicity} \quad (20)$$

When the brem gluon (with momentum  $k$ ) is emitted from the left of leg 4, the amplitudes are the same except that  $\mathbf{K}_{4,k}$  appears instead of  $\mathbf{K}_{k,4}$ . Thus the amplitudes for emission from left and right have opposite signs. The amplitudes do not cancel, however, because they have different gauge group structure. As already mentioned at  $N_c = \infty$  the two terms enter the cross section incoherently. When the brem gluon has the same helicity as leg 4 and is collinear with  $p_4$ , it and gluon 4 are distinguished only by their  $p^+$  values. Then we arbitrarily call the one with smaller  $|p^+|$  the brem gluon.

Now it is easy to see that

$$\mathbf{K}_{k,4} = -p_4^+ \hat{\mathbf{k}}, \quad (k + p_4)^2 = -p_4^+ \hat{\mathbf{k}}^2 / k^+ = -2p_4^+ \hat{k}^\wedge \hat{k}^\vee / k^+ \quad (21)$$

Then we have immediately,

$$\begin{aligned} \frac{dp_4}{2|p_4^+|} \frac{d\mathbf{k}}{2|k^+|(2\pi)^3} (|A^\vee|^2 + |A^\wedge|^2) = \\ \frac{d\mathbf{P}}{2|P^+|} \frac{d\hat{\mathbf{k}}}{|k^+|(2\pi)^3} \frac{p_4^+}{k^+ + p_4^+} \left( \frac{(k^+ + p_4^+)^2}{p_4^{+2}} + \frac{p_4^{+2}}{(k^+ + p_4^+)^2} \right) \frac{g^2}{\hat{\mathbf{k}}^2} |A_{\text{Core}}|^2 \end{aligned} \quad (22)$$

The collinear divergence is now transparent in the integration over  $\hat{\mathbf{k}}$  near zero. It is the coefficient of the phase space factor  $d\mathbf{P}/2|P^+|$  that we should compare to the square of the tree amplitudes with self-energy corrections on external lines.

We now show that the collinear divergence just isolated, when summed over all possible  $k^+$  is canceled by a corresponding divergence in the self-energy correction to leg 4. This cancellation is a consequence of the Lee-Nauenberg theorem [7], which stipulates that all collinear states with the same energy be included. The total energy of the brem gluon and the gluon represented by leg 4 is in the collinear limit

$$k^- + p_4^- = \frac{k^2}{2k^+} + \frac{p_4^2}{2p_4^+} = \frac{p_4^2}{2p_4^+} (1 + k^+/p_4^+) = \frac{P^2}{2(k^+ + p_4^+)} \quad (23)$$

which makes it clear that we should include all  $k^+, p_4^+$  consistent with fixed  $P^+ = k^+ + p_4^+$ . We temporarily regulate the divergence by giving the collinear gluons a small mass  $\mu$ . Since lightcone phase space is mass independent,  $\mu$  only appears in the above analysis in the expression for  $(k + p_4)^2$ :

$$(k + p_4)^2 \rightarrow -\frac{p_4^+}{k^+} \left( \hat{\mathbf{k}}^2 + \mu^2 \frac{(k^+ + p_4^+)^2}{p_4^{+2}} \right) \quad (24)$$

With this regulator the transverse momentum integral we need is just

$$\int_{0 < \hat{k}^2 |p_4^+| < |k^+| \Delta^2} \frac{d\hat{\mathbf{k}} \hat{k}^2}{[\hat{k}^2 + \mu^2(k^+ + p_4^+)^2/p_4^{+2}]^2} = \pi \ln \frac{|k^+| |p_4^+| \Delta^2}{|k^+ + p_4^+|^2 \mu^2 e} \quad (25)$$

Then the coefficient of the jet phase space factor is

$$\int_{\Delta} \frac{d\mathbf{k}}{2|k^+|(2\pi)^3} (|A^\vee|^2 + |A^\wedge|^2) = \frac{1}{|k^+|} \frac{g^2}{8\pi^2} \left( \frac{(k^+ + p_4^+)}{p_4^+} + \frac{p_4^{+3}}{(k^+ + p_4^+)^3} \right) |A_{\text{Core}}|^2 \ln \frac{|k^+| |p_4^+| \Delta^2}{|k^+ + p_4^+|^2 \mu^2 e} \quad (26)$$

The blowup as  $\mu \rightarrow 0$  is the collinear divergence we are seeking to resolve. According to the Lee-Nauenberg theorem, to get an infrared safe quantity we must sum over all  $k^+$  in the range  $0 < |k^+| < |P^+|$ . And we must also include brems emission from the left of leg 4. The first term represents the emission of a brems gluon with identical helicity to leg 4, so when we sum that term over the whole range of  $k^+$  we have included emission from both the left and right of leg 4. However the second term, represents the emission of a brems gluon with opposite helicity, and when summed over the whole range gives only brems gluon emission from the right of leg 4. The emission of an opposite helicity gluon (with momentum  $k$ ) from the left has the same squared amplitude, but it is convenient to switch the roles of  $k$  and  $p_4$ , so  $k$  always refers to the right gluon. Then the total emission rate is given by

$$\begin{aligned} \sum_{0 < |k^+| < |P^+|} \int_{\Delta} \frac{d\mathbf{k}}{2|k^+|(2\pi)^3} (|A^\vee|^2 + |A_R^\wedge|^2 + |A_L^\wedge|^2) = \\ \sum_{k^+} \frac{g^2}{8\pi^2} \left( \frac{|P^+|}{|k^+(P^+ - k^+)|} + \frac{|P^+ - k^+|^3}{|k^+ P^+|^3} + \frac{|k^+|^3}{|(P^+ - k^+) P^+|^3} \right) |A_{\text{Core}}|^2 \ln \frac{|k^+(P^+ - k^+)| \Delta^2}{|P^+|^2 \mu^2 e} \end{aligned} \quad (27)$$

Calling  $x = |k^+|/|P^+|$ ,  $|P^+|$  times the quantity in parentheses can be rearranged

$$\begin{aligned} \frac{1}{x(1-x)} + \frac{(1-x)^3}{x} + \frac{x^3}{1-x} &= 2 \left( x(1-x) + \frac{x}{1-x} + \frac{1-x}{x} \right) \\ &= 2 \left( \frac{|k^+| |P^+ - k^+|}{P^{+2}} + \frac{|k^+|}{|P^+ - k^+|} + \frac{|k^+|}{|P^+ - k^+|} \right) \end{aligned} \quad (28)$$

So with this notation the squared amplitude for jet production along gluon 4 is

$$\begin{aligned} \sum_{0 < |k^+| < |P^+|} \int_{\Delta} \frac{d\mathbf{k}}{2|k^+|(2\pi)^3} (|A^\vee|^2 + |A_R^\wedge|^2 + |A_L^\wedge|^2) = \\ \frac{g^2}{4\pi^2} \frac{|A_{\text{Core}}|^2}{|P^+|} \sum_{k^+} \left( x(1-x) + \frac{x}{1-x} + \frac{1-x}{x} \right) \ln \frac{x(1-x) \Delta^2}{\mu^2 e} \end{aligned} \quad (29)$$

We still have to include the self energy corrections on the external lines.

In Ref. [4] we obtained

$$\Pi^{\wedge\vee} = \frac{g^2}{4\pi^2} \frac{p^2}{|p^+|} \sum_{k^+} \left( x(1-x) + \frac{x}{1-x} + \frac{1-x}{x} \right) \ln(x(1-x)p^2 \delta e^\gamma) \quad (30)$$

for the gluon self-energy for an off-shell gluon of momentum  $p$  after subtraction of the counterterms necessary to keep the gluon massless. Redoing the calculation with a mass  $\mu$  for the particles circulating in the loop yields the modification

$$\Pi_\mu^{\wedge\vee} = \frac{g^2}{4\pi^2} \frac{1}{|p^+|} \sum_{k^+} (\mu^2 + x(1-x)p^2) \left( 1 + \frac{1}{(1-x)^2} + \frac{1}{x^2} \right) \ln((\mu^2 + x(1-x)p^2) \delta e^\gamma) \quad (31)$$

$$\rightarrow \text{Constant} + \frac{p^2}{|p^+|} \frac{g^2}{4\pi^2} \sum_{k^+} \left( x(1-x) + \frac{x}{1-x} + \frac{1-x}{x} \right) \ln(\mu^2 \delta e^{\gamma+1}) \quad (32)$$



in the on-shell limit  $p^2 \rightarrow 0$ . The constant (which is of order  $\mu^2$ ) must be absorbed in a mass counterterm to keep the gluon massless. On the external line the  $p^2$  is canceled by the extra gluon propagator, and the effect of the correction is just the wave function renormalization

$$Z = 1 + \frac{g^2}{4\pi^2} \frac{1}{|p^+|} \sum_{k^+} \left( x(1-x) + \frac{x}{1-x} + \frac{1-x}{x} \right) \ln(\mu^2 \delta e^{\gamma+1}) \quad (33)$$

Of course the tree amplitude is multiplied by  $\sqrt{Z}$  for each leg and the squared tree amplitude by a factor of  $Z$ . Thus the correction on leg 4 is just

$$(Z-1)|A_{\text{Core}}|^2 = \frac{g^2}{4\pi^2} \frac{|A_{\text{Core}}|^2}{|p^+|} \sum_{k^+} \left( x(1-x) + \frac{x}{1-x} + \frac{1-x}{x} \right) \ln(\mu^2 \delta e^{\gamma+1}) \quad (34)$$

Combining this self energy correction with (29) gives for the complete jet production, identifying  $P = p$ ,

$$\langle |\mathcal{M}|^2 \rangle_{\text{jet}} = \frac{g^2}{4\pi^2} \frac{|A_{\text{Core}}|^2}{|p^+|} \sum_{k^+} \left( x(1-x) + \frac{x}{1-x} + \frac{1-x}{x} \right) \ln[x(1-x)\Delta^2 \delta e^{\gamma}] \quad (35)$$

We see that the collinear divergence problem, which in lightcone gauge is confined to the self energy insertions on external lines, is resolved provided we interpret scattering amplitudes in terms of jets.

However, this is not the end of the story because we still see UV divergences ( $\ln \delta$ ) and IR divergences due to  $x$  near 0 or 1. As explained in [4] the latter divergences are regulated by discretization of  $p^+$ . The UV divergence from this calculation is to be combined with the UV divergences from triangle, box, and internal line self-energy diagrams to give the appropriate scale dependent coupling. Part of the IR divergences here will be canceled by soft gluon bremsstrahlung which we discuss next. In particular, we must find that the dependence on the resolution  $\Delta$  is finite. But there will be residual IR divergences that are to be canceled by IR divergences from the other one loop diagrams (box, triangle, etc.).

Our discussion of jet production only included the diagram with brem gluon attached the external leg identified with the jet (for definiteness we chose leg 4), neglecting its interference with diagrams with the brem gluon attached to other lines. This approximation is only valid, however, when the resolution is smaller than the momentum  $k^+ \mathbf{p}_4/p_4^+$  of the gluon in the jet. We can stipulate that all of the momenta  $p_i$  are of order  $O(1)$ , but even so for small enough  $k^+$  it is essential that we include the other diagrams. In this case all components of the brem gluon momentum are small and we must combine gluon emission from all legs coherently. In the large  $N_c$  limit life is simpler because we only need to include coherent emission from neighboring lines as already discussed.

For definiteness, focus first on the coherent emission of a gluon between legs 3 and 4, both of which we assume to have outgoing helicity. Then the emission amplitudes are

$$A^\vee = -2g A_{\text{Core}} \left[ \frac{k^+ + p_4^+}{k^+ p_4^+} \frac{K_{k,4}^\vee}{(k + p_4)^2} + \frac{k^+ + p_3^+}{k^+ p_3^+} \frac{K_{3,k}^\vee}{(k + p_3)^2} \right] \quad (36)$$

$$A^\wedge = -2g A_{\text{Core}} \left[ \frac{p_4^+}{k^+ (k^+ + p_4^+)} \frac{K_{k,4}^\wedge}{(k + p_4)^2} + \frac{p_3^+}{k^+ (k^+ + p_3^+)} \frac{K_{3,k}^\wedge}{(k + p_3)^2} \right] \quad (37)$$

In these formulas we have assumed that  $A_{\text{Core}}$  is the same in both terms, which is approximately true when all components of  $k$  are small. The case where  $k$  collinear with one of the external momenta but not small will not introduce errors, because in that case the interference between different terms is negligible. For this we need to insist that no two external legs are collinear, which we do. The squared amplitudes are:

$$\begin{aligned} |A^\vee|^2 &= 4g^2 |A_{\text{Core}}|^2 \left\{ \frac{(k^+ + p_4^+)^2 \mathbf{K}_{k,4}^2}{2k^{+2} p_4^{+2} (k + p_4)^4} + \frac{(k^+ + p_3^+)^2 \mathbf{K}_{k,3}^2}{2k^{+2} p_3^{+2} (k + p_3)^4} - \frac{(k^+ + p_4^+)(k^+ + p_3^+) \mathbf{K}_{k,3} \cdot \mathbf{K}_{k,4}}{k^{+2} p_3^+ p_4^+ (k + p_3)^2 (k + p_4)^2} \right\} \\ |A^\wedge|^2 &= 4g^2 |A_{\text{Core}}|^2 \left\{ \frac{p_4^{+2} \mathbf{K}_{k,4}^2}{2k^{+2} (k^+ + p_4^+)^2 (k + p_4)^4} + \frac{p_3^{+2} \mathbf{K}_{k,3}^2}{2k^{+2} (k^+ + p_3^+)^2 (k + p_3)^4} \right. \\ &\quad \left. - \frac{p_3^+ p_4^+ \mathbf{K}_{k,3} \cdot \mathbf{K}_{k,4}}{k^{+2} (k^+ + p_4^+)(k^+ + p_3^+)(k + p_3)^2 (k + p_4)^2} \right\} \end{aligned}$$

These expressions contribute to the jet cross section for jets both along  $p_3$  and  $p_4$ . Let us define  $\mathbf{v}_i \equiv \mathbf{p}_i/p_i^+$ . Then they should be integrated over the union of the two domains  $\mathcal{D}_4 : (\mathbf{k} - k^+ \mathbf{v}_4)^2 < k^+ \Delta^2/p_4^+$  and  $\mathcal{D}_3 : (\mathbf{k} - k^+ \mathbf{v}_3)^2 < k^+ \Delta^2/p_3^+$ . In order not to double count we integrate over the entire first domain, but we integrate only over the part of the second domain satisfying  $(\mathbf{k} - k^+ \mathbf{v}_4)^2 > k^+ \Delta^2/p_4^+$ . The necessary integrals over  $\mathcal{D}_4$  can be found in Appendix A. Define  $\hat{\mathbf{k}} \equiv \mathbf{k} - k^+ \mathbf{v}_4$ , and give all of the gluons 3,4 and  $k$  a small mass  $\mu$ . Then we have

$$\frac{1}{2k^{+2}} \int_{\mathcal{D}_4} d\hat{\mathbf{k}} \frac{\mathbf{K}_{k,4}^2}{(k+p_4)^4} = \frac{1}{2} \int_{\mathcal{D}_4} d\hat{\mathbf{k}} \frac{\hat{\mathbf{k}}^2}{(\hat{\mathbf{k}}^2 + \mu^2(k^+ + p_4^+)^2/p_4^{+2})^2} = \frac{\pi}{2} \ln \frac{k^+ p_4^+ \Delta^2}{(k^+ + p_4^+)^2 \mu^2 e} \quad (38)$$

$$\begin{aligned} \frac{1}{2k^{+2}} \int_{\mathcal{D}_4} d\hat{\mathbf{k}} \frac{\mathbf{K}_{k,3}^2}{(k+p_3)^4} &= \frac{1}{2} \int_{\mathcal{D}_4} d\hat{\mathbf{k}} \frac{k^{+2}(\hat{\mathbf{k}} - k^+ \mathbf{v}_{34})^2}{((\hat{\mathbf{k}} - k^+ \mathbf{v}_{34})^2 + \mu^2(k^+ + p_3^+)^2/p_3^{+2})^2} \\ &= \frac{\pi}{2} \begin{cases} \ln \frac{k^+ p_3^{+2}(\Delta^2 - k^+ p_4^+ \mathbf{v}_{34}^2)}{p_4^+(k^+ + p_3^+)^2 \mu^2 e}, & |k^+| < \frac{\Delta^2}{|p_4^+| \mathbf{v}_{34}^2} \\ \ln \frac{k^+ p_4^+ \mathbf{v}_{34}^2}{k^+ p_4^+ \mathbf{v}_{34}^2 - \Delta^2}, & |k^+| > \frac{\Delta^2}{|p_4^+| \mathbf{v}_{34}^2} \end{cases} \end{aligned} \quad (39)$$

$$\begin{aligned} \frac{1}{k^{+2}} \int_{\mathcal{D}_4} d\hat{\mathbf{k}} \frac{\mathbf{K}_{k,3} \cdot \mathbf{K}_{k,4}}{(k+p_3)^2(k+p_4)^2} &= \int_{\mathcal{D}_4} d\hat{\mathbf{k}} \frac{\hat{\mathbf{k}} \cdot (\hat{\mathbf{k}} - k^+ \mathbf{v}_{34})}{(\hat{\mathbf{k}}^2 + \mu^2(k^+ + p_4^+)^2/p_4^{+2})(\hat{\mathbf{k}} - k^+ \mathbf{v}_{34})^2 + \mu^2(k^+ + p_3^+)^2/p_3^{+2}} \\ &= \frac{\pi}{2} \begin{cases} \ln \frac{\Delta^4}{k^{+2} p_4^{+2} \mathbf{v}_{34}^4}, & |k^+| < \frac{\Delta^2}{|p_4^+| \mathbf{v}_{34}^2} \\ 0, & |k^+| > \frac{\Delta^2}{|p_4^+| \mathbf{v}_{34}^2} \end{cases} \end{aligned} \quad (40)$$

where the final forms are valid as  $\mu \rightarrow 0$ . We have used the identity

$$(k+p_i)^2 = -\frac{\mathbf{K}_{k,i}^2 + \mu^2(k^+ + p_i^+)^2}{k^+ p_i^+} = -\frac{p_i^{+2}(\mathbf{k} - k^+ \mathbf{v}_i)^2 + \mu^2(k^+ + p_i^+)^2}{k^+ p_i^+} \quad (41)$$

to simplify the integrands.

In assembling these contributions we write separate equations for small and large  $k^+$ , simplifying the coefficients in the first case:

$$\begin{aligned} \int_{\mathcal{D}_4} d\hat{\mathbf{k}} \frac{|A^\vee|^2 + |A^\wedge|^2}{16|k^+| \pi^3} &\approx \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2 |k^+|} \ln \frac{k^{+4} p_3^{+2} p_4^{+2} \mathbf{v}_{34}^4 (1 - k^+ p_4^+ \mathbf{v}_{34}^2/\Delta^2)}{(k^+ + p_4^+)^2 (k^+ + p_3^+)^2 \mu^4 e^2} \quad \text{for } |k^+| < \frac{\Delta^2}{|p_4^+| \mathbf{v}_{34}^2} \\ \int_{\mathcal{D}_4} d\hat{\mathbf{k}} \frac{|A^\vee|^2 + |A^\wedge|^2}{16|k^+| \pi^3} &= \frac{g^2 |A_{\text{Core}}|^2}{8\pi^2} \left\{ \left( \frac{(k^+ + p_4^+)^2}{p_4^{+2}} + \frac{p_4^{+2}}{(k^+ + p_4^+)^2} \right) \ln \frac{k^+ p_4^+ \Delta^2}{(k^+ + p_4^+)^2 \mu^2 e} \right. \\ &\quad \left. - \left( \frac{(k^+ + p_3^+)^2}{p_3^{+2}} + \frac{p_3^{+2}}{(k^+ + p_3^+)^2} \right) \ln \left( 1 - \frac{\Delta^2}{k^+ p_4^+ \mathbf{v}_{34}^2} \right) \right\} \quad \text{for } |k^+| > \frac{\Delta^2}{|p_4^+| \mathbf{v}_{34}^2} \\ &\approx \frac{g^2 |A_{\text{Core}}|^2}{8\pi^2} \left\{ \left( \frac{(k^+ + p_4^+)^2}{p_4^{+2}} + \frac{p_4^{+2}}{(k^+ + p_4^+)^2} \right) \ln \frac{k^+ p_4^+ \Delta^2}{(k^+ + p_4^+)^2 \mu^2 e} \right. \\ &\quad \left. - 2 \ln \left( 1 - \frac{\Delta^2}{k^+ p_4^+ \mathbf{v}_{34}^2} \right) \right\} \quad \text{for } |k^+| > \frac{\Delta^2}{|p_4^+| \mathbf{v}_{34}^2} \end{aligned} \quad (42)$$

where the approximation in the last line is valid because the logarithm factor cuts off large  $k^+$ . We still need to add the contribution of the part of the domain  $\mathcal{D}_3$  that doesn't intersect  $\mathcal{D}_4$ .

To handle the double constraint on the domain of integration, it is convenient to divide the contribution into two contributions,

$$I : \quad \frac{k^+}{p_4^+} \Delta^2 < (\mathbf{k} - k^+ \mathbf{v}_4)^2 < \frac{k^+}{p_4^+} \Delta_0^2; \quad (\mathbf{k} - k^+ \mathbf{v}_3)^2 < \frac{k^+}{p_3^+} \Delta^2 \quad (44)$$

$$II : \quad \frac{k^+}{p_4^+} \Delta_0^2 < (\mathbf{k} - k^+ \mathbf{v}_4)^2; \quad (\mathbf{k} - k^+ \mathbf{v}_3)^2 < \frac{k^+}{p_3^+} \Delta^2 \quad (45)$$

where we choose  $\Delta_0 \gg \Delta$  large enough so that in region II we only need to include the diagram with the brem gluon attached to leg 3. It is not hard to show that in region II  $k^+$  necessarily satisfies

$$k^+ > \frac{1}{v_{34}^2} \left( \frac{\Delta_0}{\sqrt{|p_4^+|}} - \frac{\Delta}{\sqrt{|p_3^+|}} \right)^2. \quad (46)$$

Unfortunately the converse is not quite true: the condition on  $k^+$  that implies  $\mathbf{k}$  is in region II is slightly more strict:

$$k^+ > \frac{1}{v_{34}^2} \left( \frac{\Delta_0}{\sqrt{|p_4^+|}} + \frac{\Delta}{\sqrt{|p_3^+|}} \right)^2. \quad (47)$$

But at least the contribution from the part of region II that satisfies this last constraint is simply given:

$$2\pi g^2 |A_{\text{Core}}|^2 \left( \frac{(k^+ + p_3^+)^2}{p_3^{+2}} + \frac{p_3^{+2}}{(k^+ + p_3^+)^2} \right) \ln \frac{k^+ p_3^+ \Delta^2}{(k^+ + p_3^+)^2 \mu^2 e}, \quad \text{for } |k^+| > \frac{1}{v_{34}^2} \left( \frac{\Delta_0}{\sqrt{|p_4^+|}} + \frac{\Delta}{\sqrt{|p_3^+|}} \right)^2 \quad (48)$$

There remains the narrow window in  $k^+$

$$\frac{1}{v_{34}^2} \left( \frac{\Delta_0}{\sqrt{|p_4^+|}} - \frac{\Delta}{\sqrt{|p_3^+|}} \right)^2 < k^+ < \frac{1}{v_{34}^2} \left( \frac{\Delta_0}{\sqrt{|p_4^+|}} + \frac{\Delta}{\sqrt{|p_3^+|}} \right)^2. \quad (49)$$

which contains a mixture of region I and II. But by taking  $\Delta_0 \gg \Delta$  the contribution of this window can be made arbitrarily small, and at the same time  $\Delta$  can be neglected in the lower limit on (48)

$$\int_{\mathcal{D}_3^{II}} d\mathbf{k} \frac{|A^\vee|^2 + |A^\wedge|^2}{16|k^+|\pi^3} \approx \frac{g^2 |A_{\text{Core}}|^2}{8\pi^2 |k^+|} \left( \frac{(k^+ + p_3^+)^2}{p_3^{+2}} + \frac{p_3^{+2}}{(k^+ + p_3^+)^2} \right) \ln \frac{k^+ p_3^+ \Delta^2}{(k^+ + p_3^+)^2 \mu^2 e}, \quad \text{for } |k^+| > \frac{\Delta_0^2}{|p_4^+| v_{34}^2} \quad (50)$$

We limit the size of  $\Delta_0$  so that all  $k^+$  contributing to region I are negligible compared to the external  $p_i^+$ . In that case the integrand for region I simplifies to

$$|A^\vee|^2 + |A^\wedge|^2 \approx 4g^2 |A_{\text{Core}}|^2 \left\{ \frac{\mathbf{K}_{k,4}^2}{k^{+2}(k+p_4)^4} + \frac{\mathbf{K}_{k,3}^2}{k^{+2}(k+p_3)^4} - 2 \frac{\mathbf{K}_{k,3} \cdot \mathbf{K}_{k,4}}{k^{+2}(k+p_3)^2(k+p_4)^2} \right\} \quad (51)$$

In region I one can show that  $\sqrt{|k^+|} > \Delta(\sqrt{|p_3^+|} - \sqrt{|p_4^+|})/\sqrt{|p_3^+ p_4^+|} v_{34}$  so we stipulate that  $|p_3^+| > |p_4^+|$  so that  $k^+$  stays away from 0 and it is safe to take the continuum limit of the  $k^+$  sums. (To deal with the case  $|p_3^+| < |p_4^+|$  we just switch the roles of legs 3 and 4 in the calculation). Now we use the identity (41) and

$$\mathbf{K}_{k,3} \cdot \mathbf{K}_{k,4} = k^{+2} p_3 \cdot p_4 - \frac{1}{2} k^+ p_3^+ (k+p_4)^2 - \frac{1}{2} k^+ p_4^+ (k+p_3)^2 - \mu^2 (p_3^+ p_4^+ + k^+ p_3^+ + k^+ p_4^+) \quad (52)$$

to simplify the integrand even further

$$\begin{aligned} |A^\vee|^2 + |A^\wedge|^2 &\approx 4g^2 |A_{\text{Core}}|^2 \left\{ \frac{-p_4^+}{k^+(k+p_4)^2} + \frac{-p_3^+ k^+ (k+p_3)^2 - \mu^2 p_3^{+2}}{k^{+2}(k+p_3)^4} \right. \\ &\quad \left. - \frac{2k^{+2} p_3 \cdot p_4 + k^+ p_3^+ (k+p_4)^2 + k^+ p_4^+ (k+p_3)^2}{k^{+2}(k+p_3)^2(k+p_4)^2} \right\} \\ &\approx 4g^2 |A_{\text{Core}}|^2 \left\{ \frac{-\mu^2 p_3^{+2}}{k^{+2}(k+p_3)^4} - \frac{2p_3 \cdot p_4}{(k+p_3)^2(k+p_4)^2} \right\} \end{aligned} \quad (53)$$

We have dropped  $\mu^2$  terms in the numerators when the denominators are prevented from vanishing strongly enough by virtue of being in region  $I$ . Since only  $(k + p_3)^2$  is allowed to get small in region  $I$ , we only needed to keep  $\mu^2$  when that factor appears squared in the denominator. The first term in braces will only receive contributions in the integration for  $\hat{k}^2 = O(\mu^2)$  as  $\mu \rightarrow 0$ . Thus the second constraint defining region  $I$  collapses to a constraint on  $k^+$  only:  $\Delta^2 < k^+ p_4^+ v_{34}^2 < \Delta_0^2$ , so the  $\hat{\mathbf{k}}$  integration can be freely done:

$$2\pi \int_0^{k^+ \Delta^2 / p_3^+} \hat{k} d\hat{k} \frac{-\mu^2}{(\hat{\mathbf{k}}^2 + \mu^2)^2} = -\pi + \pi \frac{\mu^2}{\mu^2 + k^+ \Delta^2 / p_3^+} \rightarrow -\pi = \pi \ln \frac{1}{e}, \quad \text{for } \Delta^2 < k^+ p_4^+ v_{34}^2 < \Delta_0^2 \quad (54)$$

The second term in braces is not only Lorentz invariant but only involves the variables constrained in defining region  $I$ :

$$-2k \cdot p_3 < \Delta^2, \quad \Delta^2 < -2k \cdot p_4 < \Delta_0^2 \quad (55)$$

(In this form the constraints coincide with the earlier ones only when  $\mu = 0$ . But the effect of the change is  $O(\mu^2)$  and negligible in region  $I$ .) Furthermore the measure for  $k$  integration  $dk^+ d\mathbf{k}/2|k^+| = d^4k \delta(k^2 + \mu^2)$  is also invariant. Thus it can be evaluated in any convenient frame (for instance, one in which  $\mathbf{p}_3 = \mathbf{p}_4 = 0$ .) The result is, assuming  $\mu \ll \Delta, \Delta_0 \ll p_i$ ,

$$2 \int_{\mathcal{D}_3^I} d^4k \frac{\delta(k^2 + \mu^2)}{(k + p_3)^2 (k + p_4)^2} \approx -\frac{\pi}{p_{34}^2} \ln \frac{\Delta_0^2}{\Delta^2} \ln \frac{\Delta_0 \Delta^3}{-p_{34}^2 \mu^2} \quad (56)$$

Of course we can't directly compare this to our previous results because they have not yet been integrated over  $k^+$ . But it is amusing to compare it with the integral of (50) over the missing range  $\Delta^2 < k^+ p_4^+ v_{34}^2 < \Delta_0^2$ :

$$\begin{aligned} \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2} \int \frac{d|k^+|}{|k^+|} \ln \frac{k^+ \Delta^2}{p_3^+ \mu^2 e} &\approx \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2} \ln \frac{\Delta_0^2}{\Delta^2} \left[ \ln \frac{\Delta^2}{|p_3^+| \mu^2 e} + \ln \frac{\Delta_0 \Delta}{v_{34}^2 |p_4^+|} \right] \\ &= \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2} \ln \frac{\Delta_0^2}{\Delta^2} \ln \frac{\Delta_0 \Delta^3}{-p_{34}^2 \mu^2 e} \end{aligned} \quad (57)$$

because  $v_{34}^2 = (\mathbf{v}_3 - \mathbf{v}_4)^2 = -p_{34}^2 / p_3^+ p_4^+$ . Remarkably, this integral exactly matches the entire effect from region  $I$ . The upshot is, that even though we didn't do the calculation this way, we can summarize the complete answer by quoting the following "results" for the  $\mathbf{k}$  integrals

$$\int_{\mathcal{D}_{34}} d\hat{\mathbf{k}} \frac{|A^\vee|^2 + |A^\wedge|^2}{16|k^+| \pi^3} \approx \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2 |k^+|} \ln \frac{k^{+4} v_{34}^4 (1 - k^+ p_4^+ v_{34}^2 / \Delta^2)}{\mu^4 e^2} \quad \text{for } |k^+| < \frac{\Delta^2}{|p_4^+| v_{34}^2} \quad (58)$$

$$\begin{aligned} \int_{\mathcal{D}_{34}} d\hat{\mathbf{k}} \frac{|A^\vee|^2 + |A^\wedge|^2}{16|k^+| \pi^3} &\approx \frac{g^2 |A_{\text{Core}}|^2}{8\pi^2 |k^+|} \left\{ \left( \frac{(k^+ + p_4^+)^2}{p_4^{+2}} + \frac{p_4^{+2}}{(k^+ + p_4^+)^2} \right) \ln \frac{k^+ p_4^+ \Delta^2}{(k^+ + p_4^+)^2 \mu^2 e} \right. \\ &\quad \left. + \left( \frac{(k^+ + p_3^+)^2}{p_3^{+2}} + \frac{p_3^{+2}}{(k^+ + p_3^+)^2} \right) \ln \frac{k^+ p_3^+ \Delta^2}{(k^+ + p_3^+)^2 \mu^2 e} - 2 \ln \left( 1 - \frac{\Delta^2}{k^+ p_4^+ v_{34}^2} \right) \right\} \quad \text{for } |k^+| > \frac{\Delta^2}{|p_4^+| v_{34}^2} \end{aligned} \quad (59)$$

We must also remember that in executing the sum over  $k^+$ , the phase space measure is treated differently for the jet associated with each leg. Namely, the  $k^+$  sum associated with the jet along leg  $i$  is taken holding  $k^+ + p_i^+ \equiv P_i^+$  fixed and there is an additional factor  $|p_i^+|/|k^+ + p_i^+| = |P_i^+ - k^+|/|P_i^+|$  arising from transforming  $dp_i^+ d\mathbf{p}_i/p_i^+ = (|P_i^+ - k^+|/|P_i^+|) dP_i^+ d\mathbf{P}_i/P_i^+$ . Fortunately these subtle modifications are only significant when  $k^+ = O(p_i^+)$ , i.e. for a hard bremsstrahlung whose contribution is dominated by a single diagram. Putting everything together we can write

$$\begin{aligned} |\mathcal{M}_{34}^{\text{Brem}}|^2 &= \frac{g^2}{8\pi^2} \sum_{i=3,4} |A_{\text{Core}}^i|^2 \sum_{|k^+| > \Delta^2 / |P_i^+| v_{34}^2} \frac{1}{|k^+|} \left( \frac{P_i^+}{P_i^+ - k^+} + \frac{(P_i^+ - k^+)^3}{P_i^{+3}} \right) \ln \frac{k^+ (P_i^+ - k^+) \Delta^2}{P_i^{+2} \mu^2 e} \\ &\quad + \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2} \sum_{|k^+| < \Delta^2 / |P_i^+| v_{34}^2} \frac{1}{|k^+|} \ln \frac{k^{+4} v_{34}^4}{\mu^4 e^2} \end{aligned} \quad (60)$$

where we have used the cancellation

$$\frac{g^2 |A_{\text{Core}}|^2}{4\pi^2} \int_0^1 \frac{dt}{t} \ln(1-t) - \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2} \int_1^\infty \frac{dt}{t} \ln(1-1/t) = 0 \quad (61)$$

valid after the (safe for these terms) limit of continuous  $k^+$ .

It is illuminating to rewrite (60) in a way that makes the symmetry under  $3 \leftrightarrow 4$  manifest.

$$\begin{aligned} |\mathcal{M}_{34}^{\text{Brem}}|^2 &= \frac{g^2}{8\pi^2} \sum_{i=3,4} |A_{\text{Core}}^i|^2 \sum_{|k^+|} \frac{1}{|k^+|} \left( \frac{P_i^+}{P_i^+ - k^+} + \frac{(P_i^+ - k^+)^3}{P_i^{+3}} \right) \ln \frac{k^+ (P_i^+ - k^+) \Delta^2}{P_i^{+2} \mu^2 e} \\ &\quad + \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2} \sum_{|k^+| < \Delta^2 / |P_4^+| v_{34}^2} \frac{1}{|k^+|} \ln \frac{k^{+2} \mathbf{v}_{34}^4 |P_3^+ P_4^+|}{\Delta^4} \end{aligned} \quad (62)$$

$$\begin{aligned} &\approx \frac{g^2}{8\pi^2} \sum_{i=3,4} |A_{\text{Core}}^i|^2 \sum_{|k^+|} \frac{1}{|k^+|} \left( \frac{P_i^+}{P_i^+ - k^+} + \frac{(P_i^+ - k^+)^3}{P_i^{+3}} \right) \ln \frac{k^+ (P_i^+ - k^+) \Delta^2}{P_i^{+2} \mu^2 e} \\ &\quad + \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2} \left[ \sum_{|k^+| < A} \frac{1}{|k^+|} \ln \frac{k^{+2} \mathbf{v}_{34}^4 |P_3^+ P_4^+|}{\Delta^4} - \ln \frac{\Delta^2}{A |P_4^+| v_{34}^2} \ln \frac{\Delta^2}{A |P_3^+| v_{34}^2} \right] \end{aligned} \quad (63)$$

Here we have picked  $A \gg m$ , the  $k^+$  discretization unit, and have approximated

$$\begin{aligned} \sum_{A < |k^+| < \Delta^2 / |P_4^+| v_{34}^2} \frac{1}{|k^+|} \ln \frac{k^{+2} \mathbf{v}_{34}^4 |P_3^+ P_4^+|}{\Delta^4} &\approx \int_A^{\Delta^2 / |P_4^+| v_{34}^2} \frac{dt}{t} \ln \frac{t^2 \mathbf{v}_{34}^4 |P_3^+ P_4^+|}{\Delta^4} \\ &= -\ln \frac{\Delta^2}{A |P_4^+| v_{34}^2} \ln \frac{\Delta^2}{A |P_3^+| v_{34}^2} \end{aligned} \quad (64)$$

In the form (63) the symmetry  $3 \leftrightarrow 4$  is transparent, but the finiteness of the  $\Delta$  dependence, manifest in (60), has been obscured: the coefficient of  $\ln \Delta^2$  has a small  $k^+$  divergence on the first line that is canceled by a small  $k^+$  divergence on the second line. Another advantage of (63) is that the first line just gives the 34 contribution to the production cross sections of jet 3 and jet 4 that matches our earlier discussion. In particular the  $\mu \rightarrow 0$  divergence is now transparently canceled by the wave function renormalization.

A virtually identical calculation applies to the absorption of extra gluons in the initial state by the right of leg 1 and by the left of leg 2. Outgoing brem gluons emitted between legs 1 and 2 are suppressed at  $N_c = \infty$  just as were incoming unseen gluons absorbed between legs 3 and 4 were suppressed. The result for soft gluon absorption between legs 1 and 2 is obtained from (60) by substituting 1,2 for 3,4. In this case  $k^+, p_1^+, p_2^+$  are all positive so the many absolute value signs can be dropped. We note that this treatment of the initial state uses the Lee-Nauenberg procedure as a model of incoming legs as incoming jets, so the four legs of the core process are treated in a parallel fashion. This is in contrast to the by now standard procedure of absorbing the initial state collinear divergences in the initial state parton distribution functions. This standard procedure is indeed appropriate in interpreting collider experiments, where the gluonic process describes the scattering of the hard constituents of incoming hadrons. The Lee-Nauenberg procedure we follow is more general and works even in theories, such as  $\mathcal{N} = 4$  supersymmetric Yang-Mills, in which hadron-like bound states of constituents don't form.

The situation for unseen gluon absorption and bremsstrahlung radiation on the left and right, either between legs 1 and 4 or between legs 2 and 3, is more complicated even at large  $N_c$ , because extra gluons in the initial state, the final state and both must be taken into account. This is because all these processes are now allowed at  $N_c = \infty$ . For definiteness let's focus on the region between 1 and 4: gluons emitted or absorbed by the left of leg 4 and the left of leg 1. The diagrams for emission of a single unobserved gluon are shown on the second line of Fig. 1. The squared amplitudes are

$$|A^\vee|^2 + |A^\wedge|^2 = 4g^2 |A_{\text{Core}}|^2 \left\{ \left[ \frac{P_4^{+2}}{2(P_4^+ - k^+)^2} + \frac{(P_4^+ - k^+)^2}{2P_4^{+2}} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_4)^2}{[(\mathbf{k} - k^+ \mathbf{v}_4)^2 + \mu^2 P_4^{+2} / (P_4^+ - k^+)^2]^2} \right\}$$

$$\begin{aligned}
& + \left[ \frac{P_1^{+2}}{2(P_1^+ - k^+)^2} + \frac{(P_1^+ - k^+)^2}{2P_1^{+2}} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2 P_1^{+2}/(P_1^+ - k^+)^2]^2} \\
& - \left[ \frac{P_4^+(P_1^+ - k^+)}{P_1^+(P_4^+ - k^+)} + \frac{P_1^+(P_4^+ - k^+)}{P_4^+(P_1^+ - k^+)} \right] \\
& \times \frac{(\mathbf{k} - k^+ \mathbf{v}_1) \cdot (\mathbf{k} - k^+ \mathbf{v}_4)}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2 P_1^{+2}/(P_1^+ - k^+)^2][(\mathbf{k} - k^+ \mathbf{v}_4)^2 + \mu^2 P_4^{+2}/(P_4^+ - k^+)^2]} \Big\}
\end{aligned}$$

here we have used  $P_i^+ = p_i^+ + k^+$ , expressed the  $(k + p_i)^2$  in terms of  $\mathbf{K}_{ij}$  which we have written out explicitly. The main differences with the 34 contribution are that the helicities of the two legs are opposite and  $p_1^+ > 0$  while  $p_4^+, k^+ < 0$ . Similarly with an extra unobserved soft gluon in the initial state the relevant diagrams have a gluon *absorbed* on the left of leg 1 or leg 4, but the result of the calculation is the same as for emission with the understanding that  $k^+ > 0$ .

As we have already seen in the 34 case, the interference term is negligible unless all components of  $k$  are small so we can simplify that term by neglecting  $k^+$  compared to the  $p_i^+$ . For the future discussion we also write out separately the outgoing and incoming gluon cases sending  $k \rightarrow -k$  in the outgoing case so that  $k^+ > 0$  in both cases:

$$\begin{aligned}
|A_{\text{Out}}^\vee|^2 + |A_{\text{Out}}^\wedge|^2 & \approx 4g^2 |A_{\text{Core}}|^2 \left\{ \left[ \frac{P_4^{+2}}{2(P_4^+ + k^+)^2} + \frac{(P_4^+ + k^+)^2}{2P_4^{+2}} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_4)^2}{[(\mathbf{k} - k^+ \mathbf{v}_4)^2 + \mu^2 P_4^{+2}/(P_4^+ + k^+)^2]^2} \right. \\
& + \left[ \frac{P_1^{+2}}{2(P_1^+ + k^+)^2} + \frac{(P_1^+ + k^+)^2}{2P_1^{+2}} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2 P_1^{+2}/(P_1^+ + k^+)^2]^2} \\
& \left. - \frac{2(\mathbf{k} - k^+ \mathbf{v}_1) \cdot (\mathbf{k} - k^+ \mathbf{v}_4)}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2][(\mathbf{k} - k^+ \mathbf{v}_4)^2 + \mu^2]} \right\} \quad (65)
\end{aligned}$$

$$\begin{aligned}
|A_{\text{In}}^\vee|^2 + |A_{\text{In}}^\wedge|^2 & = 4g^2 |A_{\text{Core}}|^2 \left\{ \left[ \frac{P_4^{+2}}{2(P_4^+ - k^+)^2} + \frac{(P_4^+ - k^+)^2}{2P_4^{+2}} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_4)^2}{[(\mathbf{k} - k^+ \mathbf{v}_4)^2 + \mu^2 P_4^{+2}/(P_4^+ - k^+)^2]^2} \right. \\
& + \left[ \frac{P_1^{+2}}{2(P_1^+ - k^+)^2} + \frac{(P_1^+ - k^+)^2}{2P_1^{+2}} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2 P_1^{+2}/(P_1^+ - k^+)^2]^2} \\
& \left. - \frac{2(\mathbf{k} - k^+ \mathbf{v}_1) \cdot (\mathbf{k} - k^+ \mathbf{v}_4)}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2][(\mathbf{k} - k^+ \mathbf{v}_4)^2 + \mu^2]} \right\} \quad (66)
\end{aligned}$$

Clearly simply adding in these two contributions can't be the whole story since only one of them is needed to cancel IR divergences from loops.

There is also a difficulty with the contributions for a hard unobserved gluon. A collinear divergence is present when an extra hard outgoing gluon is collinear with either leg 4 or leg 1. When collinear with leg 4 it is simply part of the jet associated with that leg and combines with the collinear gluon emission from the right of leg 4 to cancel the collinear divergence in the self energy correction to leg 4. The divergence coming from an outgoing extra gluon collinear with leg 1 has a very different meaning. Since we have stipulated that none of the gluons in the core process are collinear, this hard extra gluon is well separated from gluons 3 and 4 and therefore in principle detectable: In this case the final state is unambiguously a three gluon state. A similar situation applies when an extra hard gluon in the initial state is collinear with leg 4. These non-jet-like collinear divergences have nothing to do with self-energy corrections on external lines and must be canceled by something else.

The mechanism [7] that takes care of the doubled soft bremsstrahlung and the non-jet-like collinear divergences is shown in Fig. 2. At first glance it seems that these diagrams wouldn't be relevant, because they are either disconnected (the first diagram) or apparently higher order (the diagrams involving two extra gluons). However, when we square the sum of these diagrams, the cross terms between the first term and the remaining three contribute as connected structures which are exactly the right order  $O(g^6)$  to be comparable to the one-loop and single gluon bremsstrahlung diagrams. The reason the last two diagrams are multiplied by 1/2 is explained in Fig. 3. It is essentially the same reasoning as that for multiplying self-energy

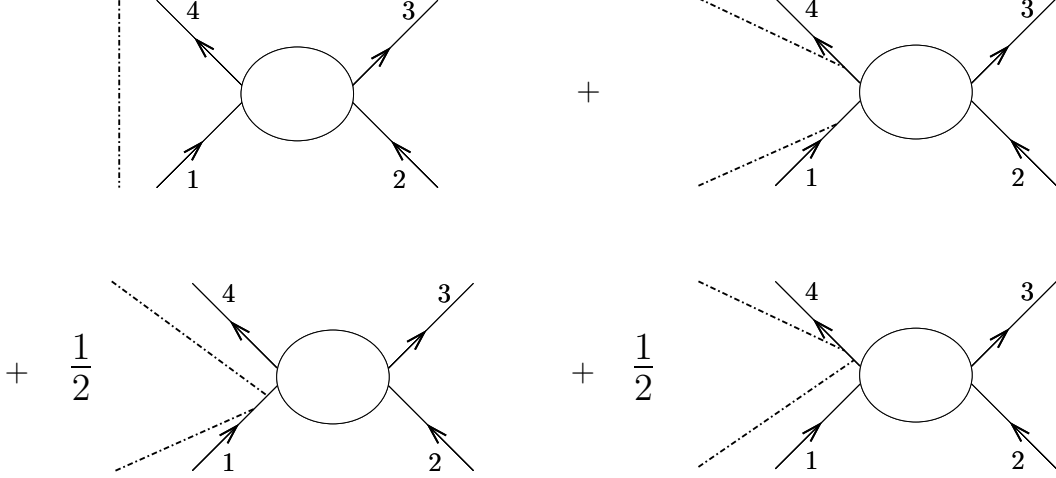


Figure 2: Diagrams representing an unseen bremsstrahlung gluon in both the initial and final state. The interference term, between the disconnected diagram on the top left and the remaining three diagrams, in the square of the sum of the four diagrams is of order  $O(g^6)$  and therefore comparable to the bremsstrahlung probability for one unseen gluon in the final state and none in the initial state or *vice versa*. To account for the factors of  $1/2$  see Fig. 3.

bubbles on external lines by  $1/2$ . Otherwise the squared amplitude would count equivalent contributions twice compared to what is required by closure for the unitary evolution operator  $U(t_1, t_2)$  or, equivalently, by unitarity of the S-matrix.

We now discuss the evaluation of the cross terms in the square of the sum of diagrams in Fig. 2. Since the gluon momentum can't simultaneously be collinear with both leg 1 and leg 4, the cross term with the second diagram on line 1 will only be singular for a soft forward scattered gluon. With incoming gluon polarization  $\wedge$  this contribution is given by

$$\begin{aligned}
 -(2g)^2 \left[ \frac{(p_4^+ - k^+)(p_1^+ + k^+)K_{4,k}^\vee K_{1,k}^\wedge}{k^{+2}p_1^+p_4^+(p_4 - k)^2(p_1 + k)^2} + \text{c.c.} \right] &= -4g^2 \frac{(p_4^+ - k^+)(p_1^+ + k^+)\mathbf{K}_{4,k} \cdot \mathbf{K}_{1,k}}{k^{+2}p_1^+p_4^+(p_4 - k)^2(p_1 + k)^2} \\
 &\approx 4g^2 \frac{(\mathbf{k} - k^+\mathbf{v}_4) \cdot (\mathbf{k} - k^+\mathbf{v}_1)}{[(\mathbf{k} - k^+\mathbf{v}_4)^2 + \mu^2][(\mathbf{k} - k^+\mathbf{v}_1)^2 + \mu^2]} \rightarrow 4g^2 \frac{(\mathbf{k} - k^+\mathbf{v}_4) \cdot (\mathbf{k} - k^+\mathbf{v}_1)}{(\mathbf{k} - k^+\mathbf{v}_4)^2(\mathbf{k} - k^+\mathbf{v}_1)^2} \quad (67)
 \end{aligned}$$

where the first form on the second line neglects  $k^+$  compared to  $p_1^+$  and  $p_4^+$ , and the second form also uses the fact that the IR divergence of this contribution is insensitive to the temporary gluon mass  $\mu$ . Although the  $p^+$  dependent factors for the other polarization of incoming gluon  $\vee$  are slightly different, this difference disappears for soft gluons. So adding the two polarizations just multiplies this result by a factor 2. Comparing this expression to (65), (66), we see that it will cancel one of the two interference terms, so only one will be counted.

Next we turn to the diagrams on the second line of Fig. 2. Since they are similar to each other, we only need do one in detail, say the first. It will give a singular contribution not only when  $k$  is soft, but also when it is hard and collinear with leg 1. In the latter case there will be three well separated gluons in the final state and therefore no confusion with the two gluon final state we want to describe. Nonetheless the collinear singularity from this contribution will cancel the one from a collinear brem gluon emission from leg 1 with no extra gluon in the initial state.

These diagrams with the forward scattering process entirely on an external leg are formally singular because the propagator between the last emission vertex and the rest of the diagram is on shell. In this regard it is analogous to self energy corrections on external lines. The most reliable way to handle such

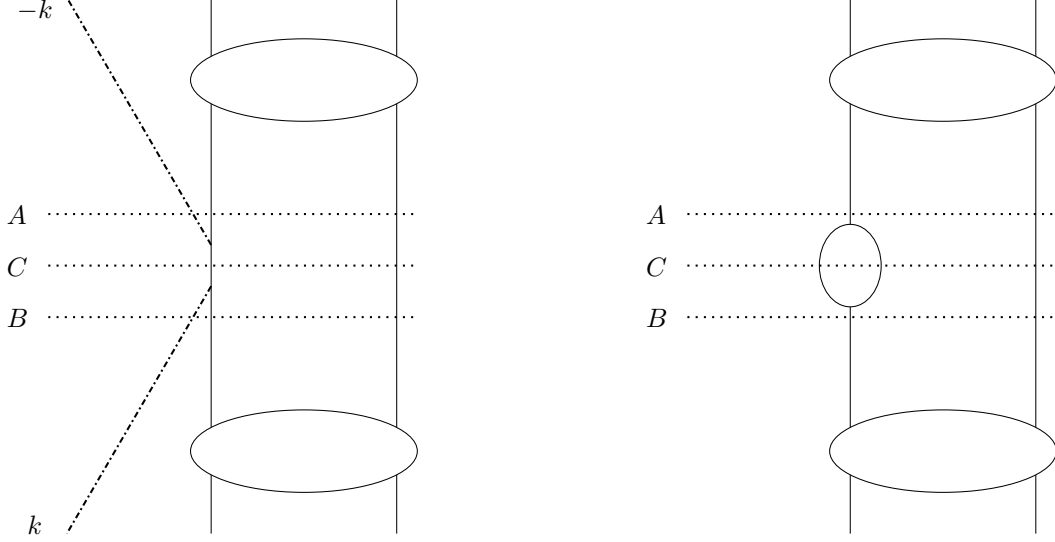


Figure 3: One of the processes on the second line of Fig. 2 viewed as a unitarity cut of a larger diagram. Because the unobserved gluon injects zero momentum into the single line it scatters from, the unitarity cuts  $A$  and  $B$  are equivalent and only one of them should be used in constructing the unitarity sum of the squared amplitude. This explains the factors of  $1/2$  in Fig. 2. The situation is entirely analogous to the well-known procedure of weighting self-energy bubbles on external lines with a factor of  $1/2$ , as indicated on the right. In both cases the unitarity cut  $C$  is distinct and unique.

situations is to compute the forward scattering process with  $p_1$  off-shell, giving the intermediate gluons a small mass  $\mu$ , and then go on shell by extracting the residue of the pole and factorizing it which means keeping only half of the correction to the residue (see Fig. 3). This process correctly discards the double pole contribution which is properly interpreted as an energy shift.<sup>7</sup>

Applying the Feynman rules to this diagram (without the factor of  $1/2$ ) with  $p_1$  off-shell and  $k^- =$

<sup>7</sup>A rougher procedure is to attempt to work on shell from the beginning and use the  $i\epsilon$ 's in the denominator to keep things finite. Then we would obtain

$$A^{\wedge\vee} = \frac{4g^2 K_{k,1}^\vee K_{k,1}^\wedge}{-i\epsilon[(p_1+k)^2 + \mu^2 - i\epsilon]} \frac{(p_1^+ + k^+)^2}{k^{+2} p_1^{+2}} A_{\text{Core}} \quad (68)$$

$$A^{\vee\wedge} = \frac{4g^2 K_{k,1}^\wedge K_{k,1}^\vee}{-i\epsilon[(p_1+k)^2 + \mu^2 - i\epsilon]} \left[ \frac{p_1^{+2}}{k^{+2}(p_1^+ + k^+)^2} + \frac{k^{+2}}{p_1^{+2}(p_1^+ + k^+)^2} \right] A_{\text{Core}} \quad (69)$$

When the cross term is constructed each of the above expressions will be multiplied by  $A_{\text{Core}}^*$  and added to its complex conjugate, and finally multiplied by  $1/2$ :

$$\begin{aligned} \text{Cross Term}^{\wedge\vee} &= \frac{1}{2} \frac{4g^2 \mathbf{K}_{k,1}^2}{-i\epsilon 2} \frac{(p_1^+ + k^+)^2}{k^{+2} p_1^{+2}} |A_{\text{Core}}|^2 \left[ \frac{1}{(p_1+k)^2 + \mu^2 - i\epsilon} - \frac{1}{(p_1+k)^2 + \mu^2 + i\epsilon} \right] \\ &= -\frac{(p_1^+ + k^+)^2}{k^{+2} p_1^{+2}} \frac{4g^2 \mathbf{K}_{k,1}^2 |A_{\text{Core}}|^2}{2[(p_1+k)^2 + \mu^2 - i\epsilon][(p_1+k)^2 + \mu^2 + i\epsilon]} \end{aligned} \quad (70)$$

$$\text{Cross Term}^{\vee\wedge} = -\left[ \frac{p_1^{+2}}{k^{+2}(p_1^+ + k^+)^2} + \frac{k^{+2}}{p_1^{+2}(p_1^+ + k^+)^2} \right] \frac{4g^2 \mathbf{K}_{k,1}^2 |A_{\text{Core}}|^2}{2[(p_1+k)^2 + \mu^2 - i\epsilon][(p_1+k)^2 + \mu^2 + i\epsilon]} \quad (71)$$

Compared to the treatment in the text, which defines jet amplitudes as they would be extracted from larger diagrams with unitarity cuts, this rough procedure misses an overall factor  $(p_1^+ + k^+)/p_1^+$ . For soft  $k$  this discrepancy is negligible, but for hard collinear  $k$  the treatment in the text is the one that reflects the Lee-Nauenberg theorem.



$(\mathbf{k}^2 + \mu^2)/2k^+$  we find

$$\begin{aligned}
\frac{A^{\wedge\vee}}{p_1^2} &= \frac{4g^2 K_{k,1}^\vee K_{k,1}^\wedge}{(p_1^2)^2[(p_1 + k)^2 + \mu^2]} \frac{(p_1^+ + k^+)^2}{k^{+2} p_1^{+2}} A_{\text{Core}}(p_1, p_2, p_3, p_4) \\
&= \frac{4g^2 (p_1^+ + k^+)^2}{k^{+2} p_1^{+2} (p_1^2)^2} \frac{-k^+ p_1^+ (\mathbf{k} - k^+ \mathbf{v}_1)^2}{2[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2 - k^+ (k^+ + p_1^+) p_1^2 / p_1^{+2}]} A_{\text{Core}}(p_1, p_2, p_3, p_4) \\
&\sim -\frac{2g^2 (p_1^+ + k^+)^2}{k^+ p_1^+ (p_1^2)^2} \left[ \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2} + \frac{p_1^2}{p_1^{+2}} \frac{k^+ (k^+ + p_1^+) (\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2]^2} \right] A_{\text{Core}} \quad (72)
\end{aligned}$$

where the superscripts indicate the polarization of the unseen gluon. The first term in square brackets is a double pole in  $p_1^2$  with a coefficient that will not have a collinear divergence (when smeared over  $\mathbf{k}$ .) Its residue will be proportional to the derivative of  $A_{\text{Core}}$  but will not contribute to the collinear divergence. The residue of only the second term in square brackets, which is a single pole in  $p_1^2$  will be divergent, and we easily read off the divergent contribution for this polarization

$$A^{\wedge\vee} \sim -\frac{2g^2 (p_1^+ + k^+)^2}{k^+ p_1^+} \frac{k^+ (k^+ + p_1^+)}{p_1^{+2}} \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2]^2} A_{\text{Core}} \quad (73)$$

The other polarization is given by

$$A^{\vee\wedge} = -2g^2 \frac{k^+ (k^+ + p_1^+)}{p_1^{+2}} \left[ \frac{p_1^{+3}}{k^+ (p_1^+ + k^+)^2} + \frac{k^{+3}}{p_1^+ (p_1^+ + k^+)^2} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2]^2} A_{\text{Core}} \quad (74)$$

There are two contributions for  $\vee\wedge$  polarization because the gluon connecting the absorption and emission vertices can have either polarization.

When the cross term is constructed each of the above expressions will be multiplied by  $A_{\text{Core}}^*$  and added to its complex conjugate, which doubles it, and finally multiplied by 1/2 which undoubles it:

$$\text{Cross Term}^{\wedge\vee} = -\frac{2g^2 (p_1^+ + k^+)^3}{p_1^{+3}} \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2]^2} |A_{\text{Core}}|^2 \quad (75)$$

$$\text{Cross Term}^{\vee\wedge} = -2g^2 \left[ \frac{p_1^+}{p_1^+ + k^+} + \frac{k^{+4}}{p_1^{+3} (p_1^+ + k^+)} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2]^2} |A_{\text{Core}}|^2 \quad (76)$$

Adding together the contribution for the two polarizations gives

$$\text{Cross Term} = -2g^2 \left[ \frac{(p_1^+ + k^+)^3}{p_1^{+3}} + \frac{p_1^+}{p_1^+ + k^+} + \frac{k^{+4}}{p_1^{+3} (p_1^+ + k^+)} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2]^2} |A_{\text{Core}}|^2 \quad (77)$$

The process described here is one where an outgoing extra gluon is collinear with gluon 1 which is incoming. This gluon is thus not in jet 3 or jet 4. It can therefore be experimentally detected unless it is too soft. Comparing to (65) we see that the first two terms of (77) almost cancel the second term of (65). Integrating the difference over a neighborhood of  $\mathbf{k} = k^+ \mathbf{v}_1$  in the limit  $\mu \rightarrow 0$  involves

$$\int d\mathbf{k} \left[ \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2 P_1^{+2} / (P_1^+ + k^+)^2]^2} - \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2]^2} \right] \sim \pi \ln \frac{P_1^{+2}}{(P_1^+ + k^+)^2} \quad (78)$$

This vanishes as  $k^+ \rightarrow 0$ , when the extra gluon is undetectable, and when  $k^+$  is finite it will be excluded from an outgoing jet along  $p_3$  or  $p_4$ . Thus it shouldn't be included in the undetected bremsstrahlung associated with the core process. The last diagram of Fig. 2 removes the divergent contribution of the first term of (66) in a similar way. The last term of (77) is negligible for soft bremsstrahlung and, when  $k^+ = O(1)$  cancels a spurious collinear divergence from a brem gluon emitted from gluon 1 with the opposite helicity to that considered here. The squared amplitude for that opposite helicity process is

$$|A'_{\text{Brem}}|^2 = 2g^2 \frac{k^{+4}}{p_1^{+2} (p_1^+ - k^+)^2} \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2 |A_{\text{Core}}(p_1 - k, p_2, p_3, p_4)|^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2 (p_1^+ - k^+)^2 / p_1^{+2}]^2} \quad (79)$$

Here the superscript denotes the polarization of the brem gluon, and  $-k$  is the incoming momentum of the brem gluon (so  $k^+ > 0$ ). Notice that the first argument of  $A_{\text{Core}}$  is  $P_1 \equiv p_1 - k$ , not  $p_1$  as in (77). To see all these cancellations, it is important to recall that  $\mathbf{k}$  is to be smeared in a narrow region about the collinear point  $\mathbf{k} = k^+ \mathbf{p}_1 / p_1^+$ . Then  $\mathbf{P}_1 \equiv \mathbf{p}_1 - \mathbf{k} \approx (p_1^+ - k^+) \mathbf{p}_1 / p_1^+$  so the integration measure of the bremsstrahlung probability is

$$\frac{d\mathbf{k}}{2|k^+|} \frac{d\mathbf{p}_1}{2p_1^+} \approx \frac{d\mathbf{k}}{2|k^+|} \frac{d\mathbf{P}_1}{2P_1^+} \frac{P_1^+ + k^+}{P_1^+} \quad (80)$$

In (77) the first argument of  $A_{\text{Core}}$  is  $p_1$  which is to be identified with  $p$  here. So we should compare (77) to  $(p^+ + k^+)/p^+$  times the appropriate term in (65).

After these cancellations what remains of the bremsstrahlung contributions (65), (66) is just

$$\begin{aligned} 4g^2 |A_{\text{Core}}|^2 & \left\{ \left[ \frac{P_4^{+2}}{2(P_4^+ + k^+)^2} + \frac{(P_4^+ + k^+)^2}{2P_4^{+2}} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_4)^2}{[(\mathbf{k} - k^+ \mathbf{v}_4)^2 + \mu^2 P_4^{+2} / (P_4^+ + k^+)^2]^2} \right. \\ & - \frac{2(\mathbf{k} - k^+ \mathbf{v}_1) \cdot (\mathbf{k} - k^+ \mathbf{v}_4)}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2][(\mathbf{k} - k^+ \mathbf{v}_4)^2 + \mu^2]} \\ & \left. + \left[ \frac{P_1^{+2}}{2(P_1^+ - k^+)^2} + \frac{(P_1^+ - k^+)^2}{2P_1^{+2}} \right] \frac{(\mathbf{k} - k^+ \mathbf{v}_1)^2}{[(\mathbf{k} - k^+ \mathbf{v}_1)^2 + \mu^2 P_1^{+2} / (P_1^+ - k^+)^2]^2} \right\} \quad (81) \end{aligned}$$

But now notice that these residual terms are the same as the 34 contribution if we identify  $P_3$  with  $-P_1$ . Thus when summed over  $k^+$  the 14 total contribution can be obtained from the 34 contribution with the substitution  $p_3 \rightarrow -p_1$ . This is consistent because unlike  $p_1 + p_4$ , which is space-like,  $p_4 - p_1$  is time-like.

To summarize this section we collect together all the contributions from hard collinear gluons, soft gluons and self energy corrections on external lines. The soft contributions boil down to a contribution like (60) for each pair of neighboring lines:

$$\begin{aligned} & \sum_{i=1}^4 \sum_{|k^+| < |P_i^+|} \frac{g^2 |A_{\text{Core}}|^2}{8\pi^2} \left( \frac{|P_i^+|}{|k^+(P_i^+ - k^+)|} + \frac{|P_i^+ - k^+|^3}{|k^+ P_i^{+3}|} + \frac{|k^+|^3}{|(P_i^+ - k^+) P_i^{+3}|} \right) \ln \frac{|k^+(P_i^+ - k^+)| \Delta^2 \delta e^\gamma}{|P_i^+|^2} \\ & + \sum_{i=1}^4 \frac{g^2 |A_{\text{Core}}|^2}{4\pi^2} \left[ \sum_{|k^+| < A} \frac{1}{|k^+|} \ln \frac{k^{+2} \mathbf{v}_{i,i+1}^4 |P_i^+ P_{i+1}^+|}{\Delta^4} - \ln \frac{\Delta^2}{A |P_i^+| v_{i,i+1}^2} \ln \frac{\Delta^2}{A |P_{i+1}^+| v_{i,i+1}^2} \right] \end{aligned}$$

Here  $v_{ij}^2 = (\mathbf{v}_i - \mathbf{v}_j)^2 = -(p_i + p_j)^2 / P_i^+ P_j^+ = |(p_i + p_j)^2| / |P_i^+ P_j^+|$ . Notice that the terms on the first line correspond to contributions associated with each leg of the diagram, whereas those on the second line involve contributions associated with pairs of consecutive lines. The first category of terms includes the wave function renormalization due to self energy bubbles on external lines

$$\sum_{i=1}^4 (Z_i - 1) |A_{\text{Core}}|^2 = \frac{g^2}{4\pi^2} \sum_{i=1}^4 \frac{|A_{\text{Core}}|^2}{|p_i^+|} \sum_{k^+} \left( x_i(1 - x_i) + \frac{x_i}{1 - x_i} + \frac{1 - x_i}{x_i} \right) \ln(\mu^2 \delta e^{\gamma+1}) \quad (82)$$

so that the collinear divergence as  $\mu \rightarrow 0$  cancels. So we see that the temporary cutoff  $\mu$  can be removed as soon as we combine everything together. We shall see that the remaining divergences in these expressions cancel against similar ones that come from the remaining one loop corrections to the glue-gluon scattering process. These include self energy insertions on internal lines together with triangle and box diagrams.

## 6 Cubic Vertex Function

We shall not include calculational details for the one loop corrections to the cubic vertex function. They can be found in [12]. Instead we present the final answers for the vertex corrections with two on-shell gluons.

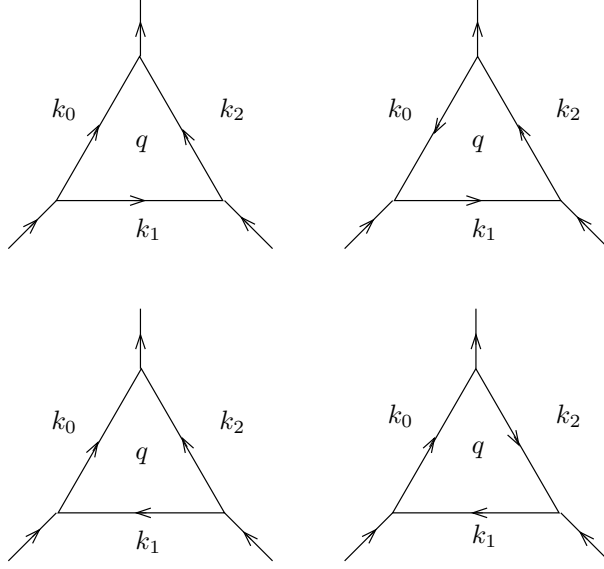


Figure 4: The triangle diagrams contributing to  $\Gamma^{^^v}$ . The labels  $q, k_0, k_1, k_2$  are dual momenta. The actual momentum of any line is the difference of the momenta of the regions it bounds.  $p_1 = k_1 - k_0$ , etc.

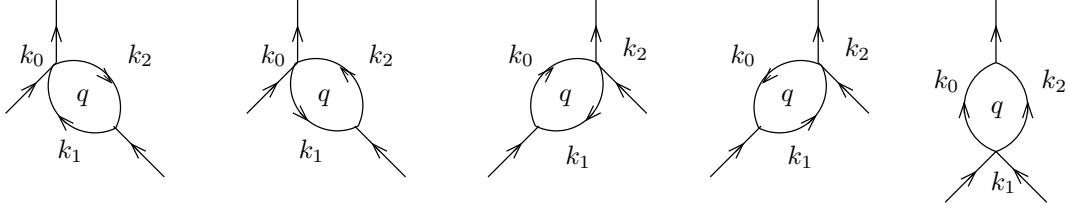


Figure 5: The swordfish diagrams contributing to  $\Gamma^{^^v}$ .

We put the combination of swordfish and triangle diagrams (see Figs. 4,5) with two like-helicities and two legs on-shell in the form

$$\Gamma_{1 \text{ loop}} + \Gamma_{\text{C.T.}} = -\frac{g^2}{8\pi^2} \Gamma_{\text{tree}} \left( \frac{70}{9} - \frac{11}{3} \ln(\delta p_o^2 e^\gamma) + S \right) + \alpha \frac{g^3}{12\pi^2} \frac{K}{p_o^+} \quad (83)$$

where the vectors  $k_i, K$  carry the polarization of the two like-helicity gluons,  $p_o$  is the four-momentum of the off-shell gluon,  $\alpha = 1$  when the on-shell gluons have like-helicity, and  $\alpha = 0$  otherwise. Finally  $S$  is an infrared sensitive term that depends on the location of the off-shell gluon, but not on any of the gluon helicities. In the case  $p_1^+, p_2^+ > 0$ , we denote by  $S_i^{q^+}(p_1, p_2)$  the value of  $S$  when leg  $i$  is off-shell, and with loop momentum chosen so that  $q^+$  is the longitudinal momentum of the internal line joining leg 1 to leg 3, satisfying  $0 < q^+ < p_{12}^+$ . Then,

$$\begin{aligned} S_1^{q^+}(p_1, p_2) &= \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right. \\ &\quad \left. + \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right. \\
& + \sum_{q^+ \neq p_1^+} \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \left. \right\} \quad (84)
\end{aligned}$$

$$\begin{aligned}
S_2^{q^+}(p_1, p_2) = & \sum_{q^+ \neq p_1^+} \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
& + \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right\} \\
& + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right. \\
& \quad \left. + \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \right\} \quad (85)
\end{aligned}$$

$$\begin{aligned}
S_3^{q^+}(p_1, p_2) = & \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_{12}^2 e^\gamma) + \ln \frac{q^+}{p_1^+ + p_2^+} \right) \right. \\
& + \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \\
& + \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \left. \right\} \\
& + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_{12}^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right) \right. \\
& + \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
& + \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \left. \right\} \quad (86)
\end{aligned}$$

In addition to these corrections to the tree level cubic vertex, the triangle diagram with three like-helicities (see Fig. 6) is non-zero, and it is given, for the case of two on-shell legs, by

$$\Gamma_{\Delta}^{\wedge\wedge\wedge} = -\frac{g^3}{6\pi^2} \frac{K^{\wedge 3}}{p_1^+ p_2^+ p_3^+ p_o^2} \quad (87)$$

$$\Gamma_{\Delta}^{\vee\vee\vee} = -\frac{g^3}{6\pi^2} \frac{K^{\vee 3}}{p_1^+ p_2^+ p_3^+ p_o^2} \quad (88)$$

where  $p_o$  is the momentum of the off-shell gluon.

## 7 Box Diagrams

For box diagrams the presence of  $q^+$  pole and double pole singularities in the integrand makes a direct evaluation of the integrals horrendous. Fortunately, it is possible to manipulate these integrands so that all of these problematic singularities reside in triangle-like diagrams. This is because on-shell tree amplitudes do not possess these singularities. We can identify tree amplitudes as sub-diagrams of one loop diagrams, but some of the legs of these sub-diagrams will be off-shell, so it would seem that features of the on-shell

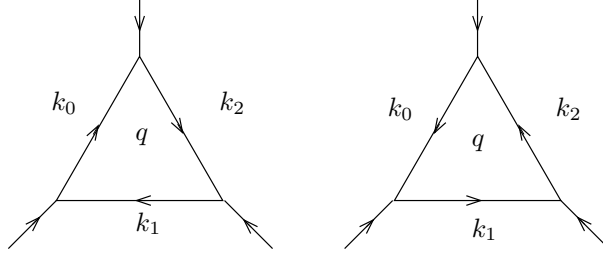


Figure 6: The triangle diagrams contributing to  $\Gamma^{^^}$ .

limit can't be exploited. However, if one leaves the denominators of the off-shell tree subdiagrams alone, then the numerators can always be written as the on-shell expression (with no  $q^+$  singularities) plus terms each of which contain at least one factor of the virtuality  $q_i^2$  of one of the off-shell legs. In a box diagram such terms will cancel a propagator reducing the required loop integrand to one with the structure of a triangle diagram. Since triangle integrals with  $q^+$  singularities are considerably easier to analyze than such box integrals, the resulting simplification is very useful. In the following subsection, we apply this technique to all of the helicity conserving box diagrams. (The helicity violating case was done in [4], where each box integrand was completely reduced to a sum of triangle-like integrands.) In the remaining subsections we complete the evaluation of the box diagrams.

## 7.1 Box Reduction

The thirteen box diagrams, seven for the helicity patterns  $\wedge \vee \wedge \vee$ , and six for  $\wedge \wedge \vee \vee$ , are shown in Figure 7 and Figure 8 respectively. The integrand of any of these box diagrams has the structure

$$\frac{1}{(2\pi)^4} \frac{R\mathcal{N}}{p_1^+ p_2^+ p_3^+ p_4^+ (q - k_0)^2 (q - k_1)^2 (q - k_2)^2 (q - k_3)^2} \quad (89)$$

where  $\mathcal{N}$  is a quartic monomial of  $K_{ij}$ 's carrying the gluon polarization information, and  $R$  is a rational function of  $q^+, p_i^+$ . There are only six possible  $\mathcal{N}$ 's:

1.  $K_{61}^\vee K_{25}^\vee K_{35}^\wedge K_{64}^\wedge$ : First diagram of Fig. 7.
2.  $K_{61}^\wedge K_{25}^\wedge K_{35}^\vee K_{64}^\vee$ : Second diagram of Fig. 7; first two diagrams of Fig. 8.
3.  $K_{61}^\vee K_{25}^\wedge K_{35}^\wedge K_{64}^\vee$ : Third diagram of Fig. 7; third diagram of Fig. 8.
4.  $K_{61}^\wedge K_{25}^\vee K_{35}^\vee K_{64}^\wedge$ : Sixth diagram of Fig. 7; sixth diagram of Fig. 8.
5.  $K_{61}^\vee K_{25}^\wedge K_{35}^\vee K_{64}^\wedge$ : Fourth diagram of Fig. 7; fourth diagram of Fig. 8.
6.  $K_{61}^\wedge K_{25}^\vee K_{35}^\wedge K_{64}^\vee$ : Fifth and seventh diagrams of Fig. 7; fifth diagram of Fig. 8.

Since these structures come in complex conjugate pairs there are really only three essentially different structures. The forms of the rational functions  $R$  change from diagram to diagram and we just list them in order:

$$\begin{aligned} & \frac{p_1^{+2} p_4^{+2}}{q^{+2} (q^+ - p_{12}^+)^2}, \quad \frac{p_2^{+2} p_3^{+2}}{q^{+2} (q^+ - p_{12}^+)^2}, \quad \frac{p_1^{+2} p_2^{+2}}{(q^+ - p_1^+)^2 (q^+ + p_4^+)^2}, \quad \frac{p_1^{+2} p_2^{+2} p_3^{+2} p_4^{+2}}{q^{+2} (q^+ - p_1^+)^2 (q^+ + p_4^+)^2 (q^+ - p_{12}^+)^2}, \\ & \frac{q^{+2} (q^+ - p_{12}^+)^2}{(q^+ - p_1^+)^2 (q^+ + p_4^+)^2}, \quad \frac{p_3^{+2} p_4^{+2}}{(q^+ - p_1^+)^2 (q^+ + p_4^+)^2}, \quad \frac{(q^+ - p_1^+)^2 (q^+ + p_4^+)^2}{q^{+2} (q^+ - p_{12}^+)^2}, \end{aligned}$$

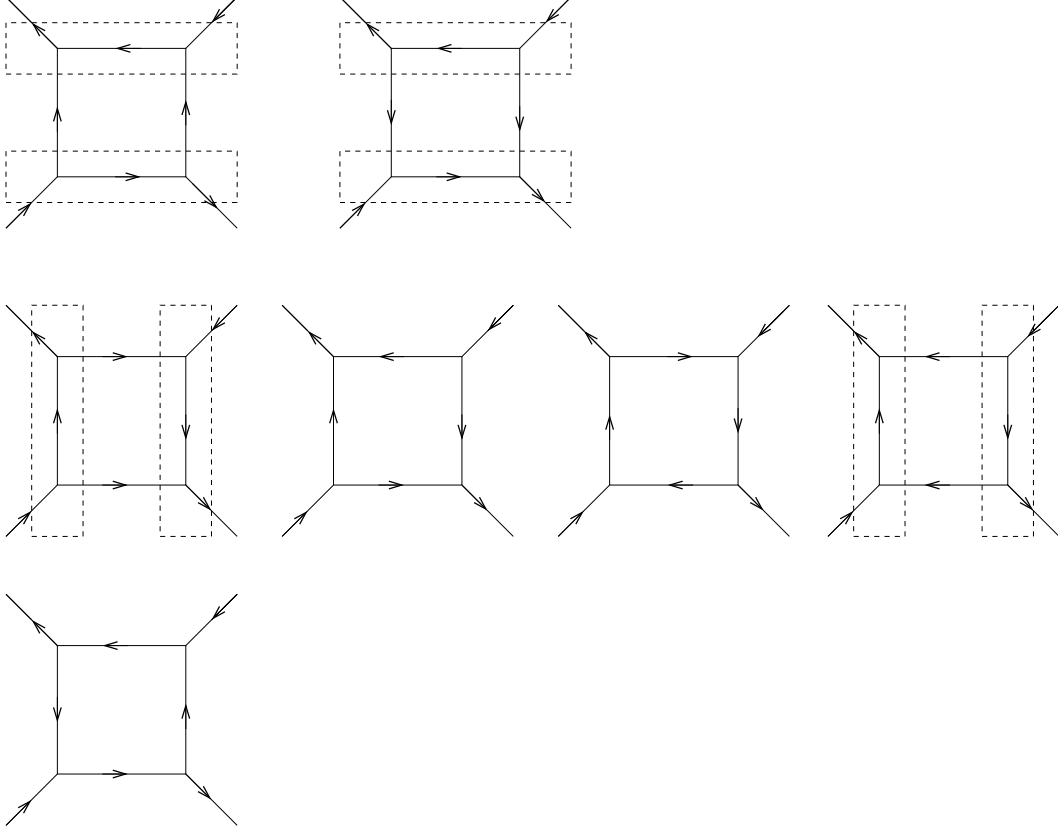


Figure 7: The boxes for the  $\wedge \vee \wedge \vee$  scattering process. The dashed boxes enclose sub diagrams whose replacement would convert the box to triangle-like loop integrals.

$$\begin{aligned}
& \frac{(q^+ - p_{12}^+)^2}{q^{+2}}, \quad \frac{q^{+2}}{(q^+ - p_{12}^+)^2}, \quad \frac{p_1^{+2} p_3^{+2}}{(q^+ - p_1^+)^2 (q^+ + p_4^+)^2}, \\
& \frac{p_1^{+2} p_4^{+2} (q^+ - p_{12}^+)^2}{q^{+2} (q^+ - p_1^+)^2 (q^+ + p_4^+)^2}, \quad \frac{p_2^{+2} p_3^{+2} q^{+2}}{(q^+ - p_1^+)^2 (q^+ + p_4^+)^2 (q^+ - p_{12}^+)^2}, \quad \frac{p_2^{+2} p_4^{+2}}{(q^+ - p_1^+)^2 (q^+ + p_4^+)^2}
\end{aligned} \tag{90}$$

We can expand each of these thirteen rational functions in partial fractions and each will then be expressed as a sum of pure double poles, pure single poles, and a constant.

$$R = C + \sum_i \left[ \frac{A_i}{(q^+ - k_i^+)^2} + \frac{B_i}{q^+ - k_i^+} \right] \tag{91}$$

where  $k_i^+$  is one of the four values  $0, p_1^+, p_{12}^+, -p_4^+$ . Except for the fourth diagram of Fig. 7, one or more of the pole terms will be absent. Also  $C = 1$  for the fifth and seventh diagrams of Fig. 7 and for the first and second diagrams of Fig. 8; and  $C = 0$  otherwise.

The eight box diagrams with a helicity violating subdiagram (enclosed by dashed boxes in the figures) can be completely reduced to triangle-like diagrams without collinear divergences. Each of these completely reducible diagrams has one of the first four polarization structures in our list, i.e. two neighboring  $K$ 's have the same polarization. Then for a like-polarization pair, one can use an identity like

$$\frac{K_{35}^\wedge K_{64}^\wedge}{q_3^2} + \frac{K_{34}^\wedge K_{65}^\wedge}{p_{12}^2} = \frac{K_{34}^\wedge K_{35}^\wedge q_0^2 + K_{64}^\wedge K_{34}^\wedge q_2^2}{p_{12}^2 q_3^2} \tag{92}$$

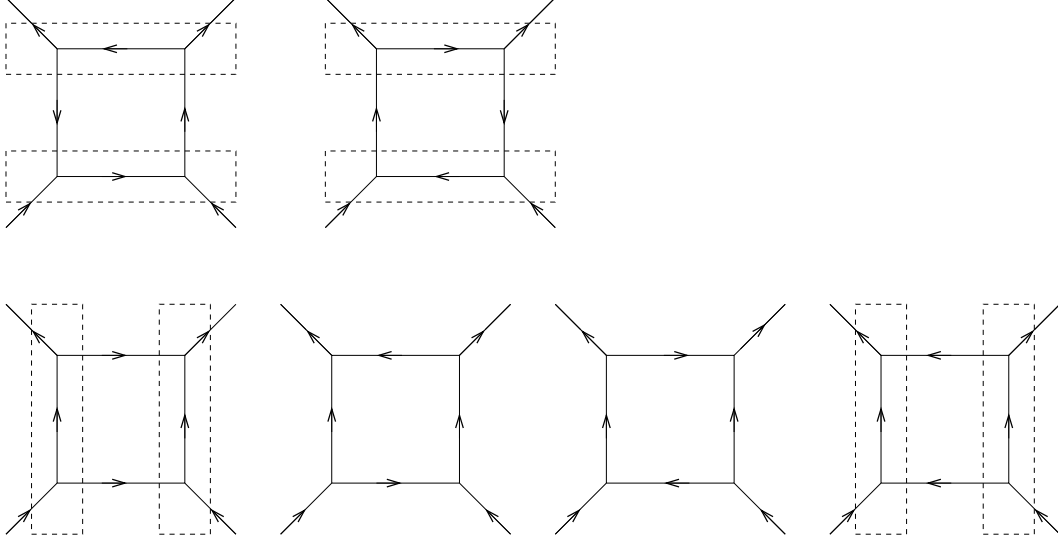


Figure 8: The boxes for the  $\wedge \wedge \vee \vee$  scattering process. The dashed boxes enclose sub-diagrams whose replacement would convert the box to triangle-like loop integrals.

which underlies the on-shell vanishing of the three like-helicity amplitude to convert the integrand to a triangle-like one which is free of collinear divergences. The integrals over  $\mathbf{q}, q^-$  can be evaluated as in [4]. The sum over  $q^+$  can be converted to an integral and carried out for the terms with no poles in  $q^+$ . The sums with  $q^+$  poles are left undone as what we call the infrared divergent contribution, and will eventually be canceled against real gluon bremsstrahlung in their contribution to cross sections. For more detail on the calculation of these eight diagrams, see Appendix E.

The remaining five diagrams cannot be completely reduced to triangle-like diagrams. But we can manipulate the integrand so that all  $q^+$  divergences are located in triangle-like diagrams. Then the remaining box integral can be straightforwardly evaluated. Since the procedure works in essentially the same way for each of these five diagrams, we shall illustrate the method by picking one of the two polarization structures, the last in our list, which appears in the fifth and seventh box diagram of Fig. 7 and in the fifth diagram of Fig. 8. As already mentioned the constant term in  $R$  only appears in these two (the fifth and seventh of Fig. 7) of the five “difficult” diagrams. Its contribution is the same for both diagrams, the integrand being

$$\frac{1}{(2\pi)^4} \frac{K_{61}^\wedge K_{25}^\vee K_{35}^\wedge K_{64}^\vee}{p_1^+ p_2^+ p_3^+ p_4^+ (q - k_0)^2 (q - k_1)^2 (q - k_2)^2 (q - k_3)^2} \quad (93)$$

and the integral of (93) is evaluated in Appendix D.

Of the pole terms in  $R$  it is sufficient for our illustrative evaluation to single out only one of the  $q^+$  pole locations, say that at  $q^+ = p_1^+$ . Thus we wish to integrate the following integrand

$$\frac{1}{(2\pi)^4} \left( \frac{A}{(q^+ - p_1^+)^2} + \frac{B}{q^+ - p_1^+} \right) \frac{K_{61}^\wedge K_{25}^\vee K_{35}^\wedge K_{64}^\vee}{p_1^+ p_2^+ p_3^+ p_4^+ q_0^2 q_1^2 q_2^2 q_3^2} \quad (94)$$

where we have introduced the shorthand notation  $q_i \equiv q - k_i$ , with  $k_i$  the dual momenta related to the actual momenta by  $p_i = k_{i+1} - k_i$ . We usually take  $k_0^\pm = 0$  but leave  $\mathbf{k}_0$  arbitrary. Next we list eight identities, the last two of which enable the desired manipulation of this model integrand.

$$\begin{aligned} K_{61}^\wedge K_{64}^\vee + K_{64}^\wedge K_{61}^\vee &= K_{61} \cdot K_{64} = \frac{q^{+2} p_{14}^2}{2} + \frac{q_0^2}{2} (p_1^+ (q^+ + p_4^+) + p_4^+ (p_1^+ - q^+)) \\ &\quad + \frac{q^+ p_4^+ q_1^2}{2} - \frac{q^+ p_1^+ q_3^2}{2} \end{aligned} \quad (95)$$

$$K_{61}^{\wedge} K_{64}^{\vee} - K_{64}^{\wedge} K_{61}^{\vee} = \frac{q^+}{p_{14}^+} (K_{14}^{\wedge} K_{61}^{\vee} - K_{14}^{\vee} K_{61}^{\wedge} + K_{14}^{\wedge} K_{64}^{\vee} - K_{14}^{\vee} K_{64}^{\wedge}) \quad (96)$$

$$K_{35}^{\wedge} K_{25}^{\vee} + K_{25}^{\wedge} K_{35}^{\vee} = K_{35} \cdot K_{25} = \frac{(p_{12}^+ - q^+)^2 p_{14}^2}{2} + \frac{q_2^2}{2} (-p_2^+ (q^+ + p_4^+) - p_3^+ (p_1^+ - q^+)) \\ + \frac{(q^+ - p_{12}^+) p_2^+ q_3^2}{2} - \frac{(q^+ - p_{12}^+) p_3^+ q_1^2}{2} \quad (97)$$

$$K_{35}^{\wedge} K_{25}^{\vee} - K_{25}^{\wedge} K_{35}^{\vee} = \frac{p_{12}^+ - q^+}{p_{23}^+} (K_{32}^{\wedge} K_{52}^{\vee} - K_{32}^{\vee} K_{52}^{\wedge} + K_{32}^{\wedge} K_{53}^{\vee} - K_{32}^{\vee} K_{53}^{\wedge}) \quad (98)$$

$$K_{35}^{\wedge} K_{64}^{\vee} + K_{64}^{\wedge} K_{35}^{\vee} = K_{35} \cdot K_{64} = \frac{(p_4^+ + q^+)^2 p_{12}^2}{2} - \frac{q_3^2}{2} (p_3^+ q^+ + p_4^+ (p_{12}^+ - q^+)) \\ + \frac{(q^+ + p_4^+) p_3^+ q_0^2}{2} - \frac{(q^+ + p_4^+) p_4^+ q_2^2}{2} \quad (99)$$

$$K_{35}^{\wedge} K_{64}^{\vee} - K_{64}^{\wedge} K_{35}^{\vee} = \frac{p_4^+ + q^+}{p_{34}^+} (K_{34}^{\wedge} K_{65}^{\vee} - K_{34}^{\vee} K_{65}^{\wedge}) \quad (100)$$

$$K_{61}^{\wedge} K_{25}^{\vee} + K_{25}^{\wedge} K_{61}^{\vee} = K_{61} \cdot K_{25} = \frac{(p_1^+ - q^+)^2 p_{12}^2}{2} + \frac{q_1^2}{2} (p_2^+ q^+ + p_1^+ (p_{12}^+ - q^+)) \\ + \frac{(p_1^+ - q^+) p_2^+ q_0^2}{2} - \frac{(p_1^+ - q^+) p_1^+ q_2^2}{2} \quad (101)$$

$$K_{61}^{\wedge} K_{25}^{\vee} - K_{25}^{\wedge} K_{61}^{\vee} = \frac{q^+ - p_1^+}{p_{12}^+} (K_{21}^{\wedge} K_{65}^{\vee} - K_{21}^{\vee} K_{65}^{\wedge}) \quad (102)$$

The first six identities are needed for other box integrands. All but the first terms of any of the right sides contain a factor of  $q_i^2$  which would cancel one of the propagators converting the box to a triangle-like diagram. Depending on which term in the partial fraction expansion we consider, we can use one of these identities to switch  $\wedge$  and  $\vee$  on a pair of  $K$  factors. For our model case we use the last two identities to write the first two  $K$  factors in two different ways

$$K_{61}^{\wedge} K_{25}^{\vee} = -K_{61}^{\vee} K_{25}^{\wedge} + \frac{(p_1^+ - q^+)^2 p_{12}^2}{2} + \frac{q_1^2}{2} (p_2^+ q^+ + p_1^+ (p_{12}^+ - q^+)) + \frac{(p_1^+ - q^+) p_2^+ q_0^2}{2} - \frac{(q^+ - p_1^+) p_1^+ q_2^2}{2} \\ K_{61}^{\wedge} K_{25}^{\vee} = K_{61}^{\vee} K_{25}^{\wedge} + \frac{q^+ - p_1^+}{p_{12}^+} (K_{21}^{\wedge} K_{65}^{\vee} - K_{21}^{\vee} K_{65}^{\wedge}) \quad (103)$$

We use the first rewrite for the double pole and the second for the single pole. The first terms of either convert the polarization structure to one which completely reduces to triangle-like diagrams free of collinear divergences just as with the eight box diagrams with helicity violating subdiagrams. Their calculation follows the models given in Appendix E. The second terms of either have an explicit factor that cancels the singular  $q^+$  denominators, leaving a box integrand free of  $q^+$  divergences. However, only the box from the second line is free of collinear divergences; its evaluation is therefore straightforward and is given in Appendix F. There are no more terms for the second rewrite, and the remaining terms of the first each have a virtuality factor  $q_0^2, q_1^2, q_2^2$  which converts the integrand to a triangle-like one. Unfortunately, both these last triangle-like diagrams and the box integrand left after the first rewrite individually have collinear divergences. They must, of course, cancel among themselves, but to achieve clean results we discuss in the next subsection a way to rearrange the integrands to finesse this difficulty.

Summarizing this subsection, we have rewritten our model integrand in the form

$$\frac{1}{(2\pi)^4} \left( \frac{A}{(q^+ - p_1^+)^2} + \frac{B}{q^+ - p_1^+} \right) \frac{K_{61}^{\wedge} K_{25}^{\vee} K_{35}^{\wedge} K_{64}^{\vee}}{p_1^+ p_2^+ p_3^+ p_4^+ q_0^2 q_1^2 q_2^2 q_3^2} = \frac{A}{(2\pi)^4} \frac{p_{12}^2 K_{35}^{\wedge} K_{64}^{\vee}}{2 p_1^+ p_2^+ p_3^+ p_4^+ q_0^2 q_1^2 q_2^2 q_3^2} \\ + \frac{B}{(2\pi)^4} \frac{(K_{21}^{\wedge} K_{65}^{\vee} - K_{21}^{\vee} K_{65}^{\wedge}) K_{35}^{\wedge} K_{64}^{\vee}}{p_{12}^+ p_1^+ p_2^+ p_3^+ p_4^+ q_0^2 q_1^2 q_2^2 q_3^2} + \text{Triangle-like} \quad (104)$$



In the next subsection we show how to deal with the collinear divergence in the first term of the right side and in the triangle-like diagrams associated with it:

$$\frac{A}{(2\pi)^4} \frac{K_{35}^\wedge K_{64}^\vee}{2(p_1^+ - q^+)^2 p_1^+ p_2^+ p_3^+ p_4^+} \left[ \frac{p_2^+ q^+ + p_1^+ (p_{12}^+ - q^+)}{q_0^2 q_2^2 q_3^2} + \frac{(p_1^+ - q^+) p_2^+}{q_1^2 q_2^2 q_3^2} - \frac{(p_1^+ - q^+) p_1^+}{q_0^2 q_1^2 q_3^2} \right] \quad (105)$$

Actually the prefactor kills the collinear divergence in the first term in the square brackets so we only need deal with the last two terms.

## 7.2 Subtracting Collinear Divergences

The box reduction we have so far accomplished has the undesirable feature that the new box integrands have collinear divergences that were not present in the original box integrands. This means that there must be canceling collinear divergences among the triangle-like diagrams that we generated in the reduction procedure. The terms with four  $K$ 's in the numerator will not have this problem but all the terms with only 2  $K$ 's in the numerator do. In order to deal with this problem we must add some triangle-like diagrams to these problematic box integrands in such a way as to regulate these divergences. To begin, we note that the terms linear in loop momentum dependent  $K$ 's can be made IR finite by a simple subtraction of two triangle-like terms. We find that each of the combinations

$$\begin{aligned} \frac{K_{25}^\vee}{q_0^2 q_1^2 q_2^2 q_3^2} - \frac{K_{12}^\vee}{q_0^2 q_1^2 p_{12}^2 q_3^2} - \frac{K_{23}^\vee}{q_0^2 q_2^2 q_3^2 p_{14}^2} &= \frac{K_{25}^\vee p_{12}^2 p_{14}^2 - K_{12}^\vee p_{14}^2 q_2^2 - K_{23}^\vee p_{12}^2 q_1^2}{q_0^2 q_1^2 q_2^2 q_3^2 p_{12}^2 p_{14}^2} \\ \frac{K_{61}^\wedge}{q_0^2 q_1^2 q_2^2 q_3^2} - \frac{K_{41}^\wedge}{q_0^2 q_2^2 p_{14}^2 q_3^2} - \frac{K_{12}^\wedge}{q_1^2 q_2^2 q_3^2 p_{12}^2} &= \frac{K_{61}^\wedge p_{12}^2 p_{14}^2 - K_{41}^\wedge p_{12}^2 q_1^2 - K_{12}^\wedge p_{14}^2 q_0^2}{q_0^2 q_1^2 q_2^2 q_3^2 p_{12}^2 p_{14}^2} \\ \frac{K_{35}^\wedge}{q_0^2 q_1^2 q_2^2 q_3^2} - \frac{K_{34}^\wedge}{q_0^2 q_1^2 p_{12}^2 q_3^2} - \frac{K_{23}^\wedge}{q_0^2 q_1^2 q_2^2 p_{14}^2} &= \frac{K_{35}^\wedge p_{12}^2 p_{14}^2 - K_{34}^\wedge p_{14}^2 q_2^2 - K_{23}^\wedge p_{12}^2 q_3^2}{q_0^2 q_1^2 q_2^2 q_3^2 p_{12}^2 p_{14}^2} \\ \frac{K_{64}^\vee}{q_0^2 q_1^2 q_2^2 q_3^2} - \frac{K_{41}^\vee}{q_0^2 q_1^2 p_{14}^2 q_2^2} - \frac{K_{34}^\vee}{q_1^2 q_2^2 q_3^2 p_{12}^2} &= \frac{K_{64}^\vee p_{12}^2 p_{14}^2 - K_{41}^\vee p_{12}^2 q_3^2 - K_{34}^\vee p_{14}^2 q_0^2}{q_0^2 q_1^2 q_2^2 q_3^2 p_{12}^2 p_{14}^2} \end{aligned} \quad (106)$$

is finite integrated in the infra-red, and the triangle-like subtractions are quadratically convergent in the ultra-violet. This nice IR behavior is not spoiled by multiplying each expression by further factors of loop momentum dependent  $K$ 's, and up to two such factors could be applied keeping the UV behavior no worse than logarithmic. Thus we can satisfactorily regulate the terms quadratic in loop momenta by simply multiplying one of these expressions by the appropriate  $K$ . There are several choices for regulating each term, so we arbitrarily choose one of them. Of course, as already mentioned the term quartic in the loop momenta is IR convergent by itself and needs no subtractions.

When we pass to the Schwinger parameterization (with the notation of Appendix C) the numerator factors in (106) enjoy a nice simplification after an appropriate shift in  $q$ :

$$\begin{aligned} K_{25}^\vee p_{12}^2 p_{14}^2 - K_{12}^\vee p_{14}^2 q_2^2 - K_{23}^\vee p_{12}^2 q_1^2 &\rightarrow (p_2^+ q^\vee - q^+ p_2^\vee) p_{12}^2 p_{14}^2 - 2q \cdot (K_0 - p_1 - p_2) K_{12}^\vee p_{14}^2 \\ &\quad - 2q \cdot (K_0 - p_1) K_{23}^\vee p_{12}^2 + (K_{23}^\vee p_{12}^2 + K_{12}^\vee p_{14}^2)(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2 - q^2) \\ K_{61}^\wedge p_{12}^2 p_{14}^2 - K_{41}^\wedge p_{12}^2 q_1^2 - K_{12}^\wedge p_{14}^2 q_0^2 &\rightarrow (p_1^+ q^\wedge - q^+ p_1^\wedge) p_{12}^2 p_{14}^2 - 2q \cdot K_0 K_{12}^\wedge p_{14}^2 - 2q \cdot (K_0 - p_1) K_{41}^\wedge p_{12}^2 \\ &\quad + (K_{41}^\wedge p_{12}^2 + K_{12}^\wedge p_{14}^2)(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2 - q^2) \\ K_{35}^\wedge p_{12}^2 p_{14}^2 - K_{34}^\wedge p_{14}^2 q_2^2 - K_{23}^\wedge p_{12}^2 q_3^2 &\rightarrow (p_3^+ q^\wedge - q^+ p_3^\wedge) p_{12}^2 p_{14}^2 - 2q \cdot (K_0 - p_1 - p_2) K_{34}^\wedge p_{14}^2 \\ &\quad - 2q \cdot (K_0 + p_4) K_{23}^\wedge p_{12}^2 + (K_{23}^\wedge p_{12}^2 + K_{34}^\wedge p_{14}^2)(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2 - q^2) \\ K_{64}^\vee p_{12}^2 p_{14}^2 - K_{41}^\vee p_{12}^2 q_3^2 - K_{34}^\vee p_{14}^2 q_0^2 &\rightarrow (p_4^+ q^\vee - q^+ p_4^\vee) p_{12}^2 p_{14}^2 - 2q \cdot K_0 K_{34}^\vee p_{14}^2 - 2q \cdot (K_0 + p_4) K_{41}^\vee p_{12}^2 \\ &\quad + (K_{41}^\vee p_{12}^2 + K_{34}^\vee p_{14}^2)(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2 - q^2) \end{aligned} \quad (107)$$

Here  $K_0 = x_2 p_1 + x_3(p_1 + p_2) - x_4 p_4$  is the  $\delta \rightarrow 0$  limit of  $K - k_0$  where  $K$  has been defined in Appendix A.

Now consider the terms quadratic in loop momentum dependent  $K$ 's. These terms involve one of the pairs  $(K_{25}, K_{35})$ ,  $(K_{35}, K_{64})$ ,  $(K_{64}, K_{61})$ ,  $(K_{61}, K_{25})$ , each pair is associated with a neighboring pair of external

lines of the box diagram. The two members of each pair have opposite polarization, with both possibilities occurring. It is easy to confirm that the terms involving each pair can be obtained from one another by cyclic symmetry. Therefore we need only analyze one class of terms, say, those involving the pair occurring in the model integrand of the previous subsection,  $K_{35}^\wedge, K_{64}^\vee$ . These terms can be regulated in the infra-red either by multiplying the last line of (106) by  $K_{35}^\wedge$  or the third line by  $K_{64}^\vee$ . We choose the former and for the other pairs make the choice dictated by cyclic symmetry. The terms in the numerator that will survive integration over  $q$  are

$$K_{35}^\wedge(K_{64}^\vee p_{12}^2 p_{14}^2 - K_{41}^\vee p_{12}^2 q_3^2 - K_{34}^\vee p_{14}^2 q_0^2) \rightarrow -2q^\wedge q^\vee (x_1 K_{34}^\wedge + x_2 K_{23}^\wedge)(K_{34}^\vee p_{14}^2 + K_{41}^\vee p_{12}^2) \\ + (x_2 K_{23}^\wedge + x_1 K_{34}^\wedge)(K_{41}^\vee p_{12}^2 + K_{34}^\vee p_{14}^2)(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2 - q^2) \quad (108)$$

In addition to dropping terms that directly integrate to 0, we also used  $-q^+ q^- \rightarrow q^\wedge q^\vee$  valid under  $q$  integration. As shown in Appendix C, the effect of  $q^2$  in the numerator is its replacement by a factor  $2H \equiv 2(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2)$  and the effect of  $q^\wedge q^\vee$  is its replacement by  $H/2$ . Thus we can replace

$$K_{35}^\wedge \left( K_{64}^\vee - \frac{K_{34}^\vee q_0^2}{p_{12}^2} - \frac{K_{41}^\vee q_3^2}{p_{14}^2} \right) \rightarrow -2(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2)(x_2 K_{23}^\wedge + x_1 K_{34}^\wedge) \left( \frac{K_{41}^\vee}{p_{14}^2} + \frac{K_{34}^\vee}{p_{12}^2} \right) \quad (109)$$

$$= -4(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2)(x_2 K_{23}^\wedge + x_1 K_{34}^\wedge) \frac{K_{13}^\wedge K_{41}^\vee K_{34}^\vee}{p_1^+ p_3^+ p_{12}^2 p_{14}^2} \quad (110)$$

The loop integral involving this pair of  $K$ 's can be done (see Appendix C)

$$\int \frac{d^4 q}{16\pi^4} \frac{p_{12}^2 K_{35}^\wedge K_{64}^\vee}{2p_1^+ p_2^+ p_3^+ p_4^+ q_0^2 q_1^2 q_2^2 q_3^2} \rightarrow I_{34} \equiv \int \frac{d^4 q}{16\pi^4} \frac{p_{12}^2 K_{35}^\wedge (K_{64}^\vee - K_{34}^\vee q_0^2/p_{12}^2 - K_{41}^\vee q_3^2/p_{14}^2)}{2p_1^+ p_2^+ p_3^+ p_4^+ q_0^2 q_1^2 q_2^2 q_3^2} \\ I_{34} = \frac{K_{13}^\wedge K_{42}^\vee}{16\pi^2 p_1^{+2} p_2^{+2} p_3^+ p_4^+} \left\{ \frac{K_{12}^\wedge K_{41}^\vee \ln(p_{14}^2/p_{12}^2)}{p_{14}^2 (p_{12}^2 + p_{14}^2)} + \frac{p_{12}^2 K_{23}^\wedge K_{41}^\vee [\pi^2 + \ln^2(p_{12}^2/p_{14}^2)]}{2p_{14}^2 (p_{12}^2 + p_{14}^2)^2} \right\} \quad (111)$$

The box integrands involving other pairs of  $K$ 's can be regulated and evaluated in an exactly parallel fashion. We shall quote the results of combining all contributions in our results Section 9.

### 7.3 Calculation of the Triangle Diagrams with Collinear Divergences

We now turn to the triangle-like diagrams which contain collinear divergences, the last two terms of (105). These divergences must be canceled when we add back in the triangle-like diagrams we subtracted from the box to cancel its collinear divergences.

$$\frac{A}{(2\pi)^4} \frac{K_{35}^\wedge K_{64}^\vee}{2(p_1^+ - q^+) p_1^+ p_2^+ p_3^+ p_4^+} \left[ \frac{p_2^+}{q_1^2 q_2^2 q_3^2} - \frac{p_1^+}{q_0^2 q_1^2 q_3^2} \right] + \frac{A}{(2\pi)^4} \frac{K_{35}^\wedge}{2p_1^+ p_2^+ p_3^+ p_4^+} \left[ \frac{p_{12}^2 K_{41}^\vee}{p_{14}^2 q_0^2 q_1^2 q_2^2} + \frac{K_{34}^\vee}{q_1^2 q_2^2 q_3^2} \right] = \\ \frac{AK_{35}^\wedge}{(2\pi)^4 2p_1^+ p_2^+ p_3^+ p_4^+} \left[ \frac{p_2^+ K_{64}^\vee + (p_1^+ - q^+) K_{34}^\vee}{(p_1^+ - q^+) q_1^2 q_2^2 q_3^2} - \frac{p_1^+ K_{64}^\vee}{(p_1^+ - q^+) q_0^2 q_1^2 q_3^2} + \frac{p_{12}^2 K_{41}^\vee}{p_{14}^2 q_0^2 q_1^2 q_2^2} \right] \quad (112)$$

The last two terms in the square brackets on the right have a collinear divergence due to the vanishing of  $q_0^2$  and  $q_1^2$  when  $q$  is collinear with  $p_1$ , specifically at  $q = q^+ p_1/p_1^+$ . In this limit we have

$$-\frac{p_1^+ K_{35}^\wedge K_{64}^\vee}{(p_1^+ - q^+) q_0^2 q_1^2 q_3^2} \rightarrow -\frac{[(p_1^+ - q^+) K_{34}^\wedge + q^+ K_{23}^\wedge] K_{41}^\vee}{(p_1^+ - q^+) q_0^2 q_1^2 p_{14}^2}, \quad \frac{p_{12}^2 K_{35}^\wedge K_{41}^\vee}{p_{14}^2 q_0^2 q_1^2 q_2^2} \rightarrow +\frac{[(p_1^+ - q^+) K_{34}^\wedge + q^+ K_{23}^\wedge] K_{41}^\vee}{(p_1^+ - q^+) q_0^2 q_1^2 p_{14}^2} \quad (113)$$

We see that this collinear divergence cancels. Similarly the first and last terms have a collinear divergence due to the vanishing of  $q_1^2$  and  $q_2^2$  when  $q_1$  is collinear with  $p_2$ , specifically at  $q = p_1 + (q^+ - p_1^+) p_2/p_2^+$ . In this limit

$$\frac{K_{35}^\wedge [p_2^+ K_{64}^\vee + (p_1^+ - q^+) K_{34}^\vee]}{(p_1^+ - q^+) q_1^2 q_2^2 q_3^2} \rightarrow \frac{(p_{12}^+ - q^+) K_{23}^\wedge K_{41}^\vee}{(p_1^+ - q^+) q_1^2 q_2^2 p_{14}^2}, \quad \frac{p_{12}^2 K_{35}^\wedge K_{41}^\vee}{p_{14}^2 q_0^2 q_1^2 q_2^2} \rightarrow -\frac{(p_{12}^+ - q^+) K_{23}^\wedge K_{41}^\vee}{(p_1^+ - q^+) q_1^2 q_2^2 p_{14}^2} \quad (114)$$

and they also cancel. The other possible collinear divergences in these expressions are killed by the polarization factors  $K_{64}^\vee$  and  $K_{35}^\vee$ .

To calculate these triangle-like contributions, we remove from each triangle structure its collinear limit. We work out one case, the first one in (113), explicitly. We integrate  $dq^-$  first<sup>8</sup>. The  $q^-$  integrals of all six types of bubble integrands and all four types of triangle-like integrands are listed in Appendix B. Note that, under the assumption that  $p_1^+ < -p_4^+$ , for  $0 < q^+ < p_1^+$  only the  $q_0$  pole contributes, for  $p_1^+ < q^+ < -p_4^+$  only the  $q_3$  pole contributes, and the integral gives 0 otherwise:

$$\begin{aligned}
& -i \int \frac{dq^-}{2\pi} \left[ -\frac{p_1^+ K_{35}^\vee K_{64}^\vee}{(p_1^+ - q^+) q_0^2 q_1^2 q_3^2} + \frac{[(p_1^+ - q^+) K_{34}^\vee + q^+ K_{23}^\vee] K_{41}^\vee}{(p_1^+ - q^+) q_0^2 q_1^2 p_{14}^2} \right] \\
& = \frac{K_{61}^\vee K_{64}^\vee K_{34}^\vee K_{41}^\vee (q^+ + p_4^+)}{p_1^+ p_4^{+2} (p_1^+ - q^+) p_{14}^2 (\mathbf{q} - q^+ \mathbf{p}_1/p_1^+)^2 (\mathbf{q} - q^+ \mathbf{p}_4/p_4^+)^2} \quad \text{for } 0 < q^+ < p_1^+ \\
& = \frac{K_{35}^\vee K_{64}^\vee q^+ p_1^+ |q^+ + p_4^+| / (q^+ - p_1^+)}{p_4^+ (\mathbf{q} - q^+ \mathbf{p}_4/p_4^+)^2 [p_4^+ (q^+ - p_1^+) (\mathbf{q} - q^+ \mathbf{p}_4/p_4^+)^2 + p_1^+ (q^+ + p_4^+) (\mathbf{q} - q^+ \mathbf{p}_1/p_1^+)^2]} \quad \text{for } p_1^+ < q^+ < -p_4^+ \\
& = \frac{K_{35}^\vee K_{64}^\vee p_1^+ |q^+ + p_4^+| / (q^+ - p_1^+)}{p_4^{+2} (\mathbf{q} - q^+ \mathbf{p}_4/p_4^+)^2 [(\mathbf{q} + \mathbf{p}_4 - (q^+ + p_4^+) \mathbf{p}_{14}/p_{14}^+)^2 - (q^+ - p_1^+) (q^+ + p_4^+) p_{14}^2/p_{14}^2]} \quad \text{for } p_1^+ < q^+ < -p_4^+
\end{aligned} \tag{115}$$

We see indeed that no collinear divergences are encountered when these expressions are integrated over  $\mathbf{q}$ .

The other cases are similar. The fact that the 012 integrand has two collinear divergences that require cancellation is not a problem, because, as seen in appendix B, after the integral over  $q^-$ , the two divergences occur in disjoint regions of  $q^+$ , so one can simply remove them additively:

$$-i \int \frac{dq^-}{2\pi} \left[ \frac{p_{12}^2 K_{35}^\vee K_{41}^\vee}{p_{14}^2 q_0^2 q_1^2 q_2^2} - \frac{[(p_1^+ - q^+) K_{34}^\vee + q^+ K_{23}^\vee] K_{41}^\vee}{(p_1^+ - q^+) q_0^2 q_1^2 p_{14}^2} + \frac{(p_{12}^+ - q^+) K_{23}^\vee K_{41}^\vee}{(p_1^+ - q^+) q_1^2 q_2^2 p_{14}^2} \right] \tag{116}$$

is thus free of collinear divergences as is

$$-i \int \frac{dq^-}{2\pi} \left[ \frac{K_{35}^\vee [p_2^+ K_{64}^\vee + (p_1^+ - q^+) K_{34}^\vee]}{(p_1^+ - q^+) q_1^2 q_2^2 q_3^2} - \frac{[(p_1^+ - q^+) K_{34}^\vee + q^+ K_{23}^\vee] K_{41}^\vee}{(p_1^+ - q^+) q_0^2 q_1^2 p_{14}^2} \right] \tag{117}$$

Another approach to the collinear divergence problem is to calculate the individual diagrams with a mass regulator  $\mu^2$ , and send  $\mu \rightarrow 0$  only after combining the terms that together are free of collinear divergences. In fact, for the purposes of automating our calculations, we found this latter regulator method more efficient, and used it to obtain the combined final results of all these integrations quoted at length in Appendix G.

## 8 Non-box One Loop Corrections to Scattering of Glue by Glue

### 8.1 Cubic Vertex Corrections and Self Energy Insertions on Internal Lines.

We quote here the contribution of triangle corrections to glue-glue scattering combined with the self energy bubbles on internal lines (see [4, 12]):

$$\begin{aligned}
\Gamma_{\Delta+\text{SE}}^{\wedge\wedge\vee\vee} &= -\frac{p_{12}^{+2} K_{34}^\vee K_{12}^\vee}{32\pi^2 p_1^+ p_2^+ p_3^+ p_4^+ p_{12}^2} \left[ \frac{11}{3} \ln(p_{12}^2 e^{\gamma\delta}) - \frac{73}{9} + \frac{p_1^+ p_2^+ + p_3^+ p_4^+}{3p_{12}^{+2}} \right. \\
&\quad \left. - S_3(p_1, p_2) - S_3(-p_4, -p_3) + \mathcal{A}(p_{12}^2, p_{12}^+) \right] \\
&\quad - \frac{1}{32\pi^2} \left( \frac{p_1^+ p_3^+}{p_2^+ p_4^+} \frac{K_{23}^\vee K_{41}^\vee}{p_{14}^{+2} p_{14}^2} + \frac{p_2^+ p_4^+}{p_1^+ p_3^+} \frac{K_{23}^\vee K_{41}^\vee}{p_{14}^{+2} p_{14}^2} \right) \left[ \frac{11}{3} \ln(p_{14}^2 e^{\gamma\delta}) - \frac{73}{9} \right. \\
&\quad \left. - S_2(-p_4, -p_{23}) - S_1(p_{14}, p_2) + \mathcal{A}(p_{14}^2, p_{14}^+) \right] \tag{118}
\end{aligned}$$

<sup>8</sup>All our formulas have treated  $d^4 q$  as Euclidean, but to integrate over  $q^-$  we convert to Minkowski measure  $d^4 q \equiv -id^4 q_M$ , and use the usual prescription for propagator denominators  $q_i^2 \rightarrow q_i^2 - i\epsilon$ . Then the integral over  $q^-$  just gives  $2\pi i$  times the sum of residues and the  $i$ 's cancel.

$$\begin{aligned}
\Gamma_{\Delta+\text{SE}}^{\wedge\vee\wedge\vee} = & -\frac{1}{32\pi^2} \left( \frac{p_1^+ p_4^+}{p_2^+ p_3^+} \frac{K_{34}^\wedge K_{12}^\vee}{p_{12}^{+2} p_{12}^2} + \frac{p_2^+ p_3^+}{p_1^+ p_4^+} \frac{K_{34}^\vee K_{12}^\wedge}{p_{12}^{+2} p_{12}^2} \right) \left[ \frac{11}{3} \ln(p_{12}^2 e^\gamma \delta) - \frac{73}{9} \right. \\
& \left. - S_3(p_1, p_2) - S_3(-p_4, -p_3) + \mathcal{A}(p_{12}^2, p_{12}^+) \right] \\
& - \frac{1}{32\pi^2} \left( \frac{p_1^+ p_2^+}{p_3^+ p_4^+} \frac{K_{23}^\wedge K_{41}^\vee}{p_{14}^{+2} p_{14}^2} + \frac{p_3^+ p_4^+}{p_1^+ p_2^+} \frac{K_{23}^\vee K_{41}^\wedge}{p_{14}^{+2} p_{14}^2} \right) \left[ \frac{11}{3} \ln(p_{14}^2 e^\gamma \delta) - \frac{73}{9} \right. \\
& \left. - S_2(-p_4, -p_{23}) - S_1(p_{14}, p_2) + \mathcal{A}(p_{14}^2, p_{14}^+) \right] \quad (119)
\end{aligned}$$

where

$$\mathcal{A}(p^2, p^+) = \sum_{q^+} \left[ \frac{1}{q^+} + \frac{1}{p^+ - q^+} \right] \ln \left\{ \frac{q^+}{p^+} \left( 1 - \frac{q^+}{p^+} \right) p^2 e^\gamma \delta \right\} \quad (120)$$

## 8.2 Quartic Triangle Diagrams

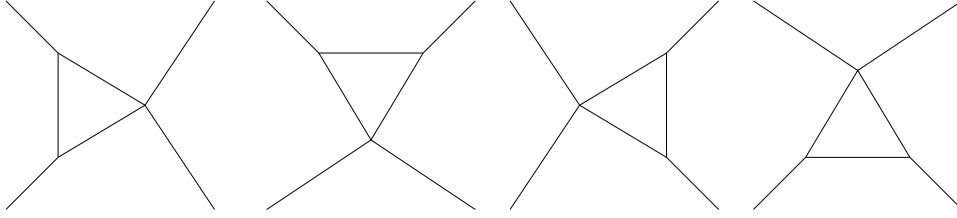


Figure 9: The quartic triangle diagrams shown generically, without arrows indicating spin flow. Particle labels 1234 are applied counter-clockwise starting at the lower left of each diagram.

There are four distinct quartic triangle structures (see Fig.9), which we label by the two legs entering the quartic vertex. Half of the diagrams for each polarization configuration are given in Figs.10,11. The integrand of each diagram has three of the four possible propagator factors  $1/q_i^2$  for  $i = 0, 1, 2, 3$ , a numerator consisting of one of the eight polarization structures  $K_{61}^\vee K_{64}^\wedge$ ,  $K_{61}^\wedge K_{64}^\vee$ ,  $K_{35}^\wedge K_{64}^\vee$ ,  $K_{35}^\vee K_{64}^\wedge$ ,  $K_{35}^\wedge K_{25}^\vee$ ,  $K_{35}^\vee K_{25}^\wedge$ ,  $K_{61}^\vee K_{25}^\wedge$ ,  $K_{61}^\wedge K_{25}^\vee$ , times a rational function of  $q^+$  and the  $p_i^+$ . The  $q^-$  and  $\mathbf{q}$  integrations are virtually identical from one diagram to the other.

A model integrand for a quartic triangle is:

$$\begin{aligned}
& -K_{6,4}^\vee K_{3,5}^\wedge \frac{1}{q_0^2} \frac{1}{q_3^2} \frac{1}{q_2^2} \\
& \frac{1}{8\pi^2} g^4 \times \\
& k_0^+ < q^+ < k_3^+ \quad \frac{1}{4} [\log H_s + \log(\delta e^\gamma)] \frac{(k_0^+ - k_3^+)(k_2^+ - q^+)}{(k_0^+ - k_2^+)} - \frac{1}{4} [\log H_d + \log(\delta e^\gamma)] (k_3^+ - q^+) \\
& k_3^+ < q^+ < k_1^+ \quad \frac{1}{4} [\log H_s + \log(\delta e^\gamma)] \frac{(-k_3^+ + k_2^+)(k_0^+ - q^+)}{(k_0^+ - k_2^+)} + \frac{1}{4} [\log H_d + \log(\delta e^\gamma)] (k_3^+ - q^+) \\
& k_1^+ < q^+ < k_2^+ \quad \frac{1}{4} [\log H_s + \log(\delta e^\gamma)] \frac{(-k_3^+ + k_2^+)(k_0^+ - q^+)}{(k_0^+ - k_2^+)} + \frac{1}{4} [\log H_d + \log(\delta e^\gamma)] (k_3^+ - q^+)
\end{aligned}$$

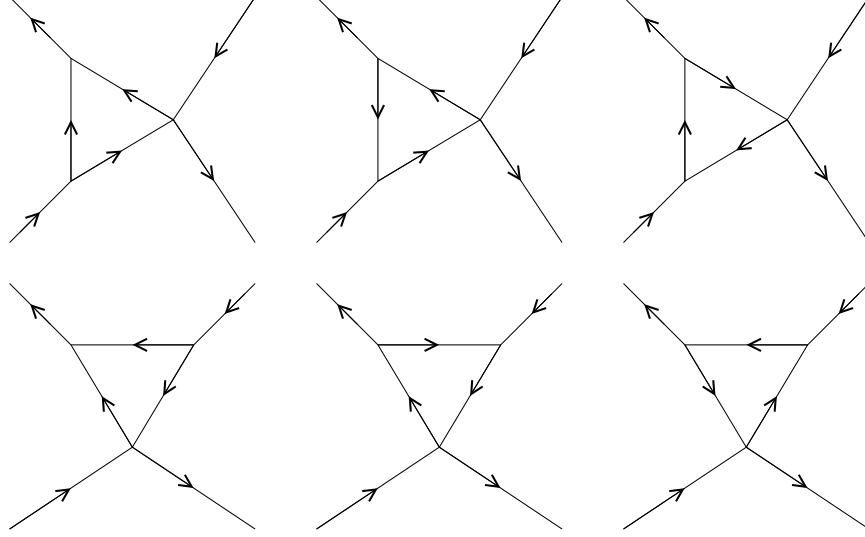


Figure 10: Half of the quartic triangle diagrams for the  $\wedge \vee \wedge \vee$  scattering process. The six others are similar but with the quartic vertex at the left or top. Particle labels 1234 are applied counter-clockwise starting at the lower left of each diagram.

The various H's are defined:

$$k_0^+ < q^+ < k_3^+$$

$$H_s = \frac{p_{12}^2(q^+ - k_0^+)(k_2^+ - q^+)}{(k_0^+ - k_2^+)^2}$$

$$H_d = -\frac{(k_3^+ - q^+)(k_0^+ - q^+)p_{12}^2}{(k_0^+ - k_3^+)(k_0^+ - k_2^+)}$$

$$H_u = \frac{p_{12}^2(k_0^+ - q^+)(-q^+ + k_1^+)}{(k_1^+ - k_0^+)(k_0^+ - k_2^+)}$$

$$H_r = -\frac{(k_0^+ - q^+)^2 p_{14}^2}{(k_1^+ - k_0^+)(k_0^+ - k_3^+)}$$

$$k_3^+ < q^+ < k_1^+$$

$$H_s = \frac{p_{12}^2(q^+ - k_0^+)(k_2^+ - q^+)}{(k_0^+ - k_2^+)^2}$$

$$H_t = -\frac{p_{14}^2(-q^+ + k_1^+)(k_3^+ - q^+)}{(k_1^+ - k_3^+)^2}$$

$$H_d = \frac{p_{12}^2(k_3^+ - q^+)(k_2^+ - q^+)}{(k_0^+ - k_2^+)(-k_3^+ + k_2^+)}$$

$$H_l = -\frac{p_{14}^2(k_2^+ - q^+)(k_3^+ - q^+)}{(-k_3^+ + k_2^+)(k_1^+ - k_3^+)}$$

$$H_u = \frac{p_{12}^2(k_0^+ - q^+)(-q^+ + k_1^+)}{(k_1^+ - k_0^+)(k_0^+ - k_2^+)}$$

$$H_r = -\frac{(k_0^+ - q^+)(-q^+ + k_1^+)p_{14}^2}{(k_1^+ - k_0^+)(k_1^+ - k_3^+)}$$

$$k_1^+ < q^+ < k_2^+$$

$$H_s = \frac{p_{12}^2(q^+ - k_0^+)(k_2^+ - q^+)}{(k_0^+ - k_2^+)^2}$$

$$H_d = \frac{p_{12}^2(k_3^+ - q^+)(k_2^+ - q^+)}{(-k_3^+ + k_2^+)(k_0^+ - k_2^+)}$$

$$H_l = -\frac{p_{14}^2(k_2^+ - q^+)^2}{(-k_2^+ + k_1^+)(-k_3^+ + k_2^+)}$$

$$H_u = -\frac{p_{12}^2(-q^+ + k_1^+)(k_2^+ - q^+)}{(-k_2^+ + k_1^+)(k_0^+ - k_2^+)}$$

Assume that the coefficient of this diagram is  $A + B(q^+ - k_3^+) + \text{single pole} + \text{double pole}$ . Multiply the above result by the prefactor, partial fraction the coefficient of the logarithm into pole terms and polynomials. The polynomials can be integrated to give::

$$\frac{1}{72}(k_3^+ - k_0^+)(k_3^+ - k_2^+)(-4Bk_3^+ + 2Bk_2^+ + 9A + 2Bk_0^+)$$

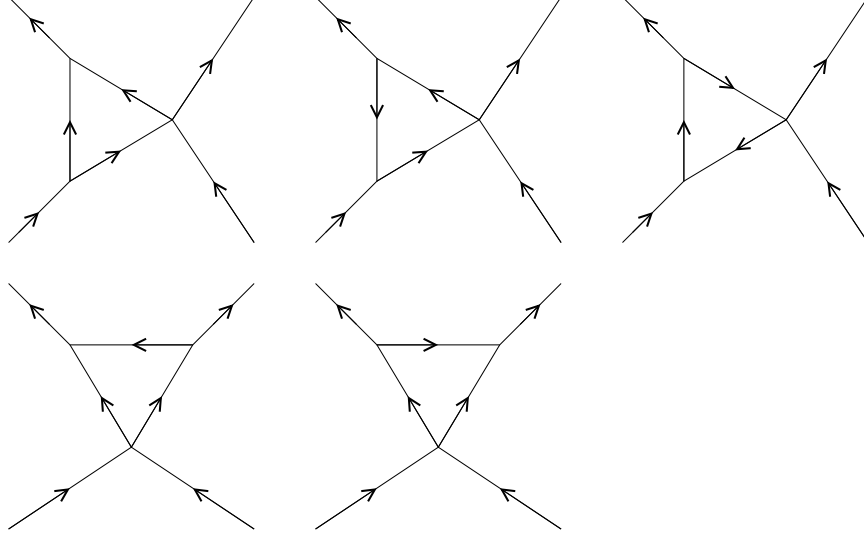


Figure 11: Half of the quartic triangle diagrams for the  $\wedge \wedge \vee \vee$  scattering process. The five others are similar but with the quartic vertex at the left or top. Particle labels 1234 are applied counter-clockwise starting at the lower left of each diagram.

$$-\frac{1}{24}(Bk_0^+ - 2Bk_3^+ + 3A + Bk_2^+)(k_3^+ - k_2^+)(k_3^+ - k_0^+) \log(p_{12}^2 \delta e^\gamma)$$

It's quite interesting that although in principle, a single pole term in the prefactor can still contribute, their net effect is zero. After the polynomial terms are integrated out, the remaining terms will be the infrared terms, they either cancel or combine with infrared terms from other diagrams into complete trees.

Similarly, the quartic triangle integrand

$$-K_{1,6}^\vee K_{5,2}^\wedge \frac{1}{q_1^2} \frac{1}{q_0^2} \frac{1}{q_2^2}$$

gives:

$$\begin{aligned} k_0^+ < q^+ < k_3^+ &+ \frac{1}{4}[\log H_s + \log(\delta e^\gamma)] \frac{(k_2^+ - q^+)(k_1^+ - k_0^+)}{(k_2^+ - k_0^+)} - \frac{1}{4}[\log H_u + \log(\delta e^\gamma)](k_1^+ - q^+) \\ k_3^+ < q^+ < k_1^+ &+ \frac{1}{4}[\log H_s + \log(\delta e^\gamma)] \frac{(k_2^+ - q^+)(k_1^+ - k_0^+)}{(k_2^+ - k_0^+)} - \frac{1}{4}[\log H_u + \log(\delta e^\gamma)](k_1^+ - q^+) \\ k_1^+ < q^+ < k_2^+ &- \frac{1}{4}[\log H_s + \log(\delta e^\gamma)] \frac{(k_1^+ - k_2^+)(-k_0^+ + q^+)}{(k_2^+ - k_0^+)} + \frac{1}{4}[\log H_u + \log(\delta e^\gamma)](k_1^+ - q^+) \end{aligned}$$

Assume that the prefactor of this integrand is  $A + B(q^+ - k_1^+) + \text{single pole} + \text{double pole}$ , a similar procedure gives for the  $A, B$  terms:

$$\begin{aligned} &\frac{1}{72}(-k_0^+ + k_1^+)(-k_2^+ + k_1^+)(-4Bk_1^+ + 2Bk_2^+ + 2Bk_0^+ + 9A) \\ &+ \frac{1}{24}(k_1^+ - k_2^+)(k_0^+ - k_1^+)(Bk_0^+ - 2Bk_1^+ + Bk_2^+ + 3A) \log(p_{12}^2 \delta e^\gamma) \end{aligned}$$

Next, the integrand

$$-K_{6,1}^\vee K_{6,4}^\wedge \frac{1}{q_1^2} \frac{1}{q_0^2} \frac{1}{q_3^2}$$

gives:

$$\begin{aligned}
k_0^+ < q^+ < k_3^+ &+ \frac{1}{4}[\log H_r + \log(\delta e^\gamma)](k_0^+ - q^+) \\
k_3^+ < q^+ < k_1^+ &- \frac{1}{4}[\log H_t + \log(\delta e^\gamma)] \frac{(k_3^+ - q^+)(k_1^+ - k_0^+)}{(k_1^+ - k_3^+)} + \frac{1}{4}[\log H_r + \log(\delta e^\gamma)](k_0^+ - q^+) \\
k_1^+ < q^+ < k_2^+ &0
\end{aligned}$$

Assume the prefactor of this model integrand is:  $A + B(q^+ - k_0^+)$  + single pole + double pole. Then the  $A, B$  terms can be integrated to give:

$$\begin{aligned}
&\frac{1}{72}(k_3^+ - k_0^+)(-k_0^+ + k_1^+)(2Bk_1^+ + 2Bk_3^+ + 9A - 4Bk_0^+) \\
&+ \frac{1}{24}(k_0^+ - k_1^+)(k_3^+ - k_0^+)(Bk_3^+ - 2Bk_0^+ + 3A + Bk_1^+) \log(p_{14}^2 \delta e^\gamma)
\end{aligned}$$

Finally, the integrand

$$-K_{3,5}^\vee K_{2,5}^\wedge \frac{1}{q_1^2} \frac{1}{q_3^2} \frac{1}{q_2^2}$$

gives

$$\begin{aligned}
k_0^+ < q^+ < k_3^+ &0 \\
k_3^+ < q^+ < k_1^+ &+ \frac{1}{4}[\log H_t + \log(\delta e^\gamma)] \frac{(-k_3^+ + k_2^+)(-q^+ + k_1^+)}{k_1^+ - k_3^+} - \frac{1}{4}[\log H_l + \log(\delta e^\gamma)](k_2^+ - q^+) \\
k_1^+ < q^+ < k_2^+ &- \frac{1}{4}[\log H_l + \log(\delta e^\gamma)](k_2^+ - q^+)
\end{aligned}$$

Assume the prefactor of this model integrand is:  $A + B(q^+ - k_2^+)$  + single pole + double pole. And the  $A, B$  terms can be integrated to give:

$$\begin{aligned}
&\frac{1}{72}(k_3^+ - k_2^+)(k_1^+ - k_2^+)(2Bk_1^+ + 2Bk_3^+ - 4k_2^+ B + 9A) \\
&- \frac{1}{24}(k_1^+ - k_2^+)(k_3^+ - k_2^+)(Bk_3^+ + Bk_1^+ + 3A - 2k_2^+ B) \log(p_{14}^2 \delta e^\gamma)
\end{aligned}$$

### 8.3 Double Quartic Diagrams

A typical double quartic integrand is:  $\frac{1}{q_1^2} \frac{1}{q_3^2}$ . This gives:

$$-\frac{1}{8\pi^2} g^4 \int_{k_3^+}^{k_1^+} \frac{dq^+}{k_1^+ - k_3^+} \frac{1}{2} \log \left[ \frac{(q^+ - k_3^+)(k_1^+ - q^+)}{(k_1^+ - k_3^+)^2} p_{14}^2 \delta e^\gamma \right]$$

We can also have a diagram like  $\frac{1}{q_0^2} \frac{1}{q_2^2}$ . Its contribution is similar, but spans over all three regions.

$$-\frac{1}{8\pi^2} g^4 \int_{k_0^+}^{k_2^+} \frac{dq^+}{k_2^+ - k_0^+} \frac{1}{2} \log \left[ \frac{(q^+ - k_0^+)(k_2^+ - q^+)}{(k_2^+ - k_0^+)^2} p_{12}^2 \delta e^\gamma \right]$$

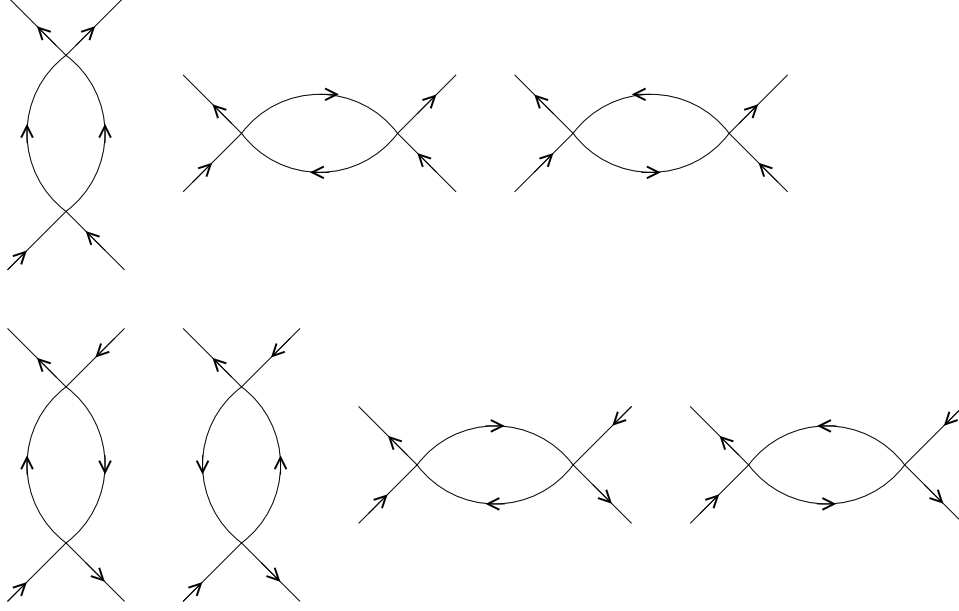


Figure 12: The double quartic diagrams for the two possible polarization patterns. Particle labels 1234 are applied counter-clockwise starting at the lower left of each diagram.

Assuming that the prefactors of the double quartic diagrams are  $A + \text{pole terms}$ . We can integrate out the  $A$  term, leaving the rest as infrared terms. Thus we have:

$$\frac{1}{8\pi^2} \frac{-1}{2} [-2 + \log(p_{14}^2 \delta e^\gamma)]$$

for the first case and

$$\frac{1}{8\pi^2} \frac{-1}{2} [-2 + \log(p_{12}^2 \delta e^\gamma)]$$

for the second case.

## 9 Final Results

In the previous sections we have described our calculational methods by choosing a single example of each distinct type, and analyzing it in detail (relegating the more tedious parts to appendices). These examples are chosen to illustrate every type of technical complication we encountered. However, along with each such example there are quite a few others involving essentially identical calculations. In fact, there are so many that we chose to automate their calculation using Matlab and Mathematica. After the results of all these many calculations are combined, there ensues a stunning simplification that allows us to present the complete elastic glue-gluon scattering amplitude in the first subsection, and in the second subsection, to obtain the complete answer for probabilities including unseen gluons in the initial and final states. This last result is compact, infrared finite, manifestly Lorentz invariant, and displays the ultraviolet behavior dictated by asymptotic freedom.



## 9.1 One Loop Corrections to Elastic Scattering of Glue by Glue

We quote here the *amputated* four gluon amplitudes, which do not include any external leg corrections.

$$\begin{aligned}
A_{\wedge\wedge\vee\vee}^{1-loop} &= -\frac{g^2}{8\pi^2} \left[ -(\log^2 \frac{p_{12}^2}{p_{14}^2} + \pi^2) - \frac{11}{3} \log[p_{14}^2 \delta e^\gamma] + \frac{73}{9} \right] A_{\wedge\wedge\vee\vee}^{tree} \\
&\quad + \frac{1}{8\pi^2} g^2 \left[ \frac{1}{3} (\mathbf{4 \text{ pt vertex}}) + g^2 \frac{2}{3} \right] + [\text{IR terms}] \\
A_{\wedge\vee\wedge\vee}^{1-loop} &= -\frac{g^2}{8\pi^2} \left[ -\frac{(p_{12}^4 + p_{12}^2 p_{14}^2 + p_{14}^4)^2}{(p_{14}^2 + p_{12}^2)^4} \left( \log^2 \frac{p_{12}^2}{p_{14}^2} + \pi^2 \right) + \frac{p_{12}^2}{3} \frac{(14p_{14}^4 + 19p_{12}^2 p_{14}^2 + 11p_{12}^4)}{(p_{14}^2 + p_{12}^2)^3} \log \frac{p_{12}^2}{p_{14}^2} \right. \\
&\quad \left. - \frac{11}{3} \log[p_{12}^2 \delta e^\gamma] + \frac{p_{14}^2 p_{12}^2}{(p_{14}^2 + p_{12}^2)^2} + \frac{73}{9} \right] A_{\wedge\vee\wedge\vee}^{tree} \\
&\quad + \frac{g^2}{8\pi^2} \left[ \frac{1}{3} (\mathbf{4 \text{ pt vertex}}) + g^2 \frac{2}{3} \right] + [\text{IR terms}]
\end{aligned}$$

The infrared terms for both helicity arrangements are the same multiples of the corresponding trees:

$$\begin{aligned}
\Gamma_{IR} &= -\frac{1}{8\pi^2} g^2 A^{tree} \times \\
&\quad k_0^+ < q^+ < k_3^+ \\
&\quad \left[ \frac{1}{q^+ - k_3^+} + \frac{1}{q^+ - k_1^+} \right] \log \frac{(k_2^+ - q^+)(-k_0^+ + q^+)p_{12}^2 \delta e^\gamma}{(k_0^+ - k_2^+)^2} \\
&\quad - \left[ \frac{2}{q^+ - k_1^+} \right] \log \frac{(k_1^+ - q^+)(q^+ - k_0^+)p_{12}^2 \delta e^\gamma}{(k_1^+ - k_0^+)(k_2^+ - k_0^+)} \\
&\quad - \left[ \frac{2}{q^+ - k_3^+} \right] \log \frac{(k_3^+ - q^+)(-k_0^+ + q^+)p_{12}^2 \delta e^\gamma}{(k_3^+ - k_0^+)(k_2^+ - k_0^+)} \\
&\quad + \left[ \frac{2}{q^+ - k_0^+} \right] \log \frac{(-k_0^+ + q^+)^2 p_{14}^2 \delta e^\gamma}{(k_1^+ - k_0^+)(k_3^+ - k_0^+)} \\
&\quad k_3^+ < q^+ < k_1^+ \\
&\quad - \left[ \frac{1}{q^+ - k_3^+} - \frac{1}{q^+ - k_1^+} \right] \log \frac{(k_2^+ - q^+)(-k_0^+ + q^+)p_{12}^2 \delta e^\gamma}{(k_0^+ - k_2^+)^2} \\
&\quad - \left[ \frac{1}{q^+ - k_0^+} - \frac{1}{q^+ - k_2^+} \right] \log \frac{(k_3^+ - q^+)(q^+ - k_1^+)p_{14}^2 \delta e^\gamma}{(-k_1^+ + k_3^+)^2} \\
&\quad - \left[ \frac{2}{q^+ - k_1^+} \right] \log \frac{(k_0^+ - q^+)(q^+ - k_1^+)p_{12}^2 \delta e^\gamma}{(-k_1^+ + k_0^+)(k_0^+ - k_2^+)} \\
&\quad + \left[ \frac{2}{q^+ - k_3^+} \right] \log \frac{(q^+ - k_3^+)(k_2^+ - q^+)p_{12}^2 \delta e^\gamma}{(k_2^+ - k_0^+)(k_2^+ - k_3^+)} \\
&\quad - \left[ \frac{2}{q^+ - k_2^+} \right] \log \frac{(-k_2^+ + q^+)(k_3^+ - q^+)p_{14}^2 \delta e^\gamma}{(k_3^+ - k_2^+)(-k_1^+ + k_3^+)} \\
&\quad + \left[ \frac{2}{q^+ - k_0^+} \right] \log \frac{(k_0^+ - q^+)(q^+ - k_1^+)p_{14}^2 \delta e^\gamma}{(-k_1^+ + k_0^+)(-k_1^+ + k_3^+)} \\
&\quad k_1^+ < q^+ < k_2^+ \\
&\quad - \left[ \frac{1}{q^+ - k_3^+} + \frac{1}{q^+ - k_1^+} \right] \log \frac{(k_2^+ - q^+)(-k_0^+ + q^+)p_{12}^2 \delta e^\gamma}{(k_0^+ - k_2^+)^2}
\end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{2}{q^+ - k_1^+} \right] \log \frac{(q^+ - k_1^+)(-k_2^+ + q^+)p_{12}^2 \delta e^\gamma}{(k_2^+ - k_1^+)(k_0^+ - k_2^+)} \\
& + \left[ \frac{2}{q^+ - k_3^+} \right] \log \frac{(q^+ - k_3^+)(k_2^+ - q^+)p_{12}^2 \delta e^\gamma}{(k_2^+ - k_0^+)(k_2^+ - k_3^+)} \\
& - \left[ \frac{2}{q^+ - k_2^+} \right] \log \frac{(-k_2^+ + q^+)^2 p_{14}^2 \delta e^\gamma}{(k_2^+ - k_1^+)(k_2^+ - k_3^+)}
\end{aligned}$$

The reader will note that the infrared sensitive terms depend on the ultraviolet cutoff  $\delta$ . This entangling of infrared and ultraviolet divergences is a familiar consequence of the way we have cut off  $p^+ = 0$  singularities. These entangled divergences are precisely cancelled by similar divergences in the self-energy corrections to external lines which contribute to the  $\prod \sqrt{Z_i}$  factors that convert the amputated amplitudes to properly normalized scattering amplitudes. When these factors are included (as they will be in the next subsection on physical probabilities), the net coefficient of  $\ln \delta$  becomes  $-11g^2/24\pi^2$  in precise agreement with asymptotic freedom [8, 9].

The terms in these amplitudes that are not multiplied by trees are Lorentz violating anomalies that must be removed by counterterms. In Section 10 we show how this can be done locally in target space and described locally on the worldsheet. As we shall see, after these counterterms are taken into account, the only change in the rest of the formula is a change  $73/9 \rightarrow 67/9$  in the constant terms multiplying the respective trees. We assume these changes have been done in our discussion of unseen gluons in the following subsection.

The expressions for the loop amplitudes are real in the unphysical region for scattering in the 12 channel where  $p_{12}^2 = -s$  and  $p_{14}^2 = -t$  are both positive. The physical region for this process is  $s > 0, t < 0$ , which we can obtain by analytic continuation. Since the physical  $s$  is above the cut on the positive real axis, we obtain the physical amplitudes by substituting  $p_{12}^2 = e^{-i\pi}s$  in the above formulas. In this way we see that in the physical region the amplitudes acquire an imaginary part. In the next subsection we use these physical region amplitudes in the calculation of gluon detection probabilities.

## 9.2 Probabilities Including Bremsstrahlung and Unseen Initial Gluons

In this section, we will focus on the case when the extra gluon (we can break the Bose symmetry by defining it to be the softest one) is between particle 3 and 4. Recalling the results of section 5, there is a total of three terms that contribute to the infrared and collinear singularity.

$$\begin{aligned}
M_{coll} &= \frac{g^2}{8\pi^2} \sum_{i=3,4} |A_{core}^i|^2 \int_0^{|P_i^+|} dk^+ \\
&\quad \left( \frac{|P_i^+|}{k^+ (|P_i^+| - k^+)} + \frac{(|P_i^+| - k^+)^3}{|k^+ P_i^{+3}|} + \frac{k^{+3}}{(|P_i^+| - k^+)|P_i^{+3}|} \right) \log \frac{k^+ (|P_i^+| - k^+) \Delta^2}{P_i^{+2}} \\
M_{soft \text{ brem}} &= \frac{g^2}{4\pi^2} |A_{core}|^2 \left[ \int_{|k^+| < A} \frac{dk^+}{|k^+|} \log \frac{k^{+2} s^2}{|P_3^+ P_4^+| \Delta^4} - \log \frac{\Delta^2 |P_3^+|}{As} \log \frac{\Delta^2 |P_4^+|}{As} \right] \\
&= \frac{g^2}{4\pi^2} |A_{core}|^2 \left[ \int_{|k^+| < A} \frac{dk^+}{|k^+|} \log \frac{k^{+4} s^2}{|P_3^+ P_4^+|^2} + \int_{|k^+| < A} \frac{dk^+}{|k^+|} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} \right. \\
&\quad \left. - \left( \log \frac{\Delta^2 |P_3^+|}{s} - \log A \right) \left( \log \frac{\Delta^2 |P_4^+|}{s} - \log A \right) \right]
\end{aligned}$$

While the infrared terms from loop calculation in region 34 is:

$$M_{loop} = \frac{-g^2 |A_{core}|^2}{4\pi^2} \left[ \int_0^{|P_4^+|} \frac{dk^+}{k^+} \log \frac{k^{+2} s}{|P_4^+| |P_3^+|} + \int_0^{|P_3^+|} \frac{dk^+}{k^+} \log \frac{k^{+2} s}{|P_3^+| |P_4^+|} \right]$$

$$+ \int_{-|P_4^+|}^{|P_3^+|} \frac{dk^+}{k^+} \log \frac{(|P_3^+| - k^+)|P_4^+|}{(|P_4^+| + k^+)|P_3^+|}]$$

First, notice that  $M_{coll}$  is organized according to which leg the collinear emission is attached, but for the sake of the current discussion, we need to break them up into different regions. For example, rewrite the term (take  $i = 3$ )

$$\frac{|P_3^+|}{k^+ (|P_3^+| - k^+)} + \frac{(|P_3^+| - k^+)^3}{k^+ |P_3^+|^3} + \frac{k^{+3}}{(|P_3^+| - k^+) |P_3^+|^3}$$

as

$$\frac{1}{k^+} + \frac{(|P_3^+| - k^+)^3}{|k^+ P_3^{+3}|} + \frac{1}{|P_3^+| - k^+} + \frac{k^{+3}}{(|P_3^+| - k^+) |P_3^{+3}|}$$

The first two term are related to the last two by substituting  $k^+ \rightarrow (P_3^+ - k^+)$ . The first two will be divergent when the extra gluon becomes soft. The last two will be divergent when the extra gluon becomes dominating over gluon 3, so, by the definition of 'the extra gluon' given above, gluon 3 becomes the 'extra one'. Hence we assign the last two terms to region 23.

Second combine  $M_{loop}$  with  $M_{soft \text{ brem}}$  (leaving out a common factor  $\frac{g^2}{8\pi^2} |A_{core}|^2$ )

$$\begin{aligned} M_{loop} + M_{soft \text{ brem}} &= -2 \left[ \int_A^{|P_4^+|} \frac{dk^+}{k^+} \log \frac{k^{+2}s}{|P_4^+||P_3^+|} + \int_A^{|P_3^+|} \frac{dk^+}{k^+} \log \frac{k^{+2}s}{|P_3^+||P_4^+|} + \mathcal{I}_{34} \right] \\ &+ 2 \int_{|k^+| < A} \frac{dk^+}{|k^+|} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} - 2 \left( \log \frac{\Delta^2 |P_3^+|}{s} - \log A \right) \left( \log \frac{\Delta^2 |P_4^+|}{s} - \log A \right) \end{aligned}$$

where  $\mathcal{I}_{34}$  is

$$\int_{-|P_4^+|}^{|P_3^+|} \frac{dk^+}{k^+} \log \frac{(|P_3^+| - k^+)|P_4^+|}{(|P_4^+| + k^+)|P_3^+|}$$

We can see that the first two terms in  $M_{soft \text{ brem}}$  cancel the divergence in the first two terms of  $M_{loop}$ . Performing the first two integrals:

$$\begin{aligned} &- 2 \left[ \log^2 |P_3^+| + \log^2 |P_4^+| - 2 \log^2 A + \log \frac{|P_3^+ P_4^+|}{A^2} \log \frac{s}{|P_3^+ P_4^+|} + \mathcal{I}_{34} \right] \\ &+ 2 \int_{0 < k^+ < A} \frac{dk^+}{k^+} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} - 2 \left( \log \frac{\Delta^2 |P_3^+|}{s} - \log A \right) \left( \log \frac{\Delta^2 |P_4^+|}{s} - \log A \right) \end{aligned}$$

Rewrite the divergent integral as

$$\begin{aligned} &2 \int_{0 < k^+ < A} \frac{dk^+}{k^+} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} \\ &= \int_0^{|P_3^+|} \frac{dk^+}{k^+} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} + \int_0^{|P_4^+|} \frac{dk^+}{k^+} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} \\ &- \left[ -\log^2 |P_3^+| - \log^2 |P_4^+| + 2 \log^2 A + \log \frac{|P_3^+ P_4^+|}{A^2} \log \frac{|P_3^+ P_4^+|}{\Delta^4} \right] \end{aligned}$$

The first two integrals will be cancelled by  $M_{coll}$  later. We get:

$$M_{loop} + M_{soft \text{ brem}} = -2 \left[ \log^2 |P_3^+| + \log^2 |P_4^+| + \log |P_3^+ P_4^+| \log \frac{s}{|P_3^+ P_4^+|} + \mathcal{I}_{34} \right]$$

$$\begin{aligned}
& + \int_0^{|P_3^+|} \frac{dk^+}{k^+} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} + \int_0^{|P_4^+|} \frac{dk^+}{k^+} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} \\
& + \left[ \log^2 |P_3^+| + \log^2 |P_4^+| - \log |P_3^+ P_4^+| \log \frac{|P_3^+ P_4^+|}{\Delta^4} \right] - 2 \log \frac{\Delta^2 |P_3^+|}{s} \log \frac{\Delta^2 |P_4^+|}{s} \\
& = -2\mathcal{I}_{34} + \int_0^{|P_3^+|} \frac{dk^+}{k^+} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} + \int_0^{|P_4^+|} \frac{dk^+}{k^+} \log \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} - 2 \log^2 \frac{\Delta^2}{s}
\end{aligned}$$

$$\begin{aligned}
M_{coll} &= \int_0^{|P_3^+|} \frac{dk^+}{k^+} \left( 1 + \frac{(|P_3^+| - k^+)^3}{|P_3^{+3}|} \right) \log \frac{k^+ (|P_3^+| - k^+) \Delta^2}{P_3^{+2}} + 3 \leftrightarrow 4 \\
&= \int_0^{|P_3^+|} \frac{dk^+}{k^+} \left( 2 + \frac{(|P_3^+| - k^+)^3 - |P_3^{+3}|}{|P_3^{+3}|} \right) \log \frac{k^+ (|P_3^+| - k^+) \Delta^2}{P_3^{+2}} + 3 \leftrightarrow 4
\end{aligned}$$

Putting everything together, we get:

$$\begin{aligned}
& -2\mathcal{I}_{34} - 2 \log^2 \frac{\Delta^2}{s} + \left\{ \int_0^{|P_3^+|} \frac{dk^+}{k^+} \log \left[ \frac{|P_3^+ P_4^+|}{k^{+2} \Delta^4} \frac{k^+ (|P_3^+| - k^+)^2 \Delta^4}{P_3^{+4}} \right] + 3 \leftrightarrow 4 \right\} \\
& + \left\{ \int_0^{|P_3^+|} \frac{dk^+}{k^+} \frac{(|P_3^+| - k^+)^3 - |P_3^{+3}|}{|P_3^{+3}|} \log \frac{k^+ (|P_3^+| - k^+) \Delta^2}{P_3^{+2}} + 3 \leftrightarrow 4 \right\} \\
& = -2\mathcal{I}_{34} - 2 \log^2 \frac{\Delta^2}{s} + \left\{ \int_0^{|P_3^+|} \frac{dk^+}{k^+} \left[ \log \frac{|P_4^+|}{|P_3^+|} + \log \frac{(|P_3^+| - k^+)^2}{P_3^{+2}} \right] + 3 \leftrightarrow 4 \right\} + 2 \left[ \frac{67}{18} - \frac{11}{6} \log \Delta^2 \right] \\
& = -2\mathcal{I}_{34} - 2 \log^2 \frac{\Delta^2}{s} + \left( -\frac{2}{3} \pi^2 \right) - \log^2 \frac{P_3^+}{P_4^+} + 2 \left[ \frac{67}{18} - \frac{11}{6} \log \Delta^2 \right] \\
& = -2 \left( -\frac{1}{2} \pi^2 - \frac{1}{2} \log^2 \frac{P_3^+}{P_4^+} \right) - 2 \log^2 \frac{\Delta^2}{s} + \left( -\frac{2}{3} \pi^2 \right) - \log^2 \frac{|P_3^+|}{|P_4^+|} + 2 \left[ \frac{67}{18} - \frac{11}{6} \log \Delta^2 \right] \\
& = -2 \log^2 \frac{\Delta^2}{s} + \frac{1}{3} \pi^2 + \left[ \frac{67}{9} - \frac{11}{3} \log \Delta^2 \right]
\end{aligned}$$

Thus, after including the (12), (23), and (41) cases, the total probabilities are:

$$\begin{aligned}
P_{\wedge\wedge\vee\vee} &= |A_{\wedge\wedge\vee\vee}|^2 \left[ 1 + \frac{g^2}{4\pi^2} \left[ -2 \log^2 \frac{\Delta^2}{s} - 2 \log^2 \frac{\Delta^2}{|t|} + 2 \cdot \frac{\pi^2}{3} + \frac{67}{9} \right. \right. \\
&\quad \left. \left. - \frac{11}{3} \left[ \log (\Delta^2 \delta e^\gamma) + \log \frac{\Delta^2}{|t|} \right] + \log^2 \frac{s}{|t|} \right] \right] \quad (121)
\end{aligned}$$

$$\begin{aligned}
P_{\wedge\vee\wedge\vee} &= |A_{\wedge\vee\wedge\vee}|^2 \left[ 1 + \frac{g^2}{4\pi^2} \left[ -2 \log^2 \frac{\Delta^2}{s} - 2 \log^2 \frac{\Delta^2}{|t|} + 2 \cdot \frac{\pi^2}{3} + \frac{67}{9} - \frac{11}{3} \left[ \log (\Delta^2 \delta e^\gamma) + \frac{1}{2} \log \frac{\Delta^4}{s|t|} \right] \right. \right. \\
&\quad \left. \left. + \frac{(s^2 + st + t^2)^2}{(t+s)^4} \log^2 \frac{s}{|t|} + \frac{(5st^2 - 5s^2t + 11t^3 - 11s^3)}{6(t+s)^3} \cdot \log \frac{s}{|t|} - \frac{ts}{(t+s)^2} \right] \right] \quad (122)
\end{aligned}$$

We see that all IR divergences have cancelled, that the UV divergences are exactly those dictated by asymptotic freedom, and Lorentz invariance is manifest.

## 10 Worksheet Description of Counterterms

As we have seen, counterterms that are polynomials in the dual momenta must be added to the two, three and four point functions in order achieve the correct Lorentz covariant results. Specifically in [4] we required

the following counterterms:

$$\Pi_{\text{C.T.}}^{\wedge\wedge} = -\frac{g^2}{12\pi^2}[k_0^{\wedge 2} + k_0^{\wedge}k_1^{\wedge} + k_1^{\wedge 2}] \quad (123)$$

$$\Pi_{\text{C.T.}}^{\wedge\vee} = -\frac{g^2}{24m\delta}p^+ + \frac{g^2}{4\pi^2\delta} \quad (124)$$

$$\Gamma_{\text{C.T.}}^{\wedge\vee\vee} = \frac{g^3}{12\pi^2}[k_0^{\wedge} + k_1^{\wedge} + k_2^{\wedge}] \quad (125)$$

which are all polynomials in the dual momenta.

Quartic counterterms must also be included in the list. At first sight the terms in  $\Gamma$  that need to be canceled seem to be rational functions of the  $p_i^+$  (which would be nonlocal in  $x^-$ ):

$$\Gamma_{\text{anom}}^{\wedge\wedge\vee\vee} = \frac{g^2}{24\pi^2} \left[ -2g^2 \frac{p_1^+ p_3^+ + p_2^+ p_4^+}{(p_1^+ + p_4^+)^2} + 2g^2 \right] \quad (126)$$

$$\Gamma_{\text{anom}}^{\wedge\vee\wedge\vee} = \frac{g^2}{24\pi^2} \left[ 2g^2 \frac{p_1^+ p_2^+ + p_3^+ p_4^+}{(p_1^+ + p_4^+)^2} + 2g^2 \frac{p_1^+ p_4^+ + p_2^+ p_3^+}{(p_1^+ + p_2^+)^2} + 2g^2 \right] \quad (127)$$

Local counterterms must be polynomials in the momenta. However, we note that the nonpolynomial parts of these anomalies are proportional to the quartic vertex contributions to the corresponding tree amplitudes. The addition of a term to  $\Pi_{\text{C.T.}}^{\wedge\vee}$ , proportional to  $p^2$ , which is an allowed counterterm by power counting, would contribute a term proportional to the part of the same trees built from pairs of cubic vertices. By tuning the coefficient of  $p^2$  we can convert these nonpolynomial anomalies to complete trees, obviating the need to cancel them with a counterterm. They just correspond to a perfectly allowed *finite* coupling renormalization. The change in  $\Pi_{\text{C.T.}}^{\wedge\vee}$ , which accomplishes this is just

$$\Pi_{\text{C.T.}}^{\wedge\vee} \rightarrow -\frac{g^2}{24m\delta}p^+ + \frac{g^2}{4\pi^2\delta} + \frac{g^2}{24\pi^2}p^2 \quad (128)$$

After this rearrangement we see that the required quartic counterterms are simply constants:

$$\Gamma_{\text{C.T.}}^{\wedge\wedge\vee\vee} = -\frac{g^4}{12\pi^2}, \quad \Gamma_{\text{C.T.}}^{\wedge\vee\wedge\vee} = -\frac{g^4}{12\pi^2} \quad (129)$$

It is noteworthy that these quartic counterterms are spin independent. This is consistent with the interpretation of the anomalies as ultraviolet artifacts of box diagrams: The large momentum behavior of box integrands must of necessity be of the form  $q^{\mu_1} q^{\mu_2} q^{\mu_3} q^{\mu_4} / q^8$  which integrates to completely symmetrized Kronecker deltas.

This is all quite satisfactory from the field theoretic point of view, but the worldsheet description makes more stringent requirements on the counterterms: They must be generated by purely local changes to the worldsheet action. Worldsheet locality is quite independent of field theoretic (target space) locality, and we must still show how it can be preserved. We start with  $\Pi_{\text{C.T.}}^{\wedge\wedge}$ . Interpreted as a contribution to the worldsheet path integral representation of a gluon propagator, it should be multiplied by  $T/2p^+$  [4].

$$\frac{T\Pi_{\text{C.T.}}^{\wedge\wedge}}{2p^+} = -\frac{g^2}{16\pi^2} \int d\tau \frac{q^{\wedge 2}(0) + q^{\wedge 2}(p^+)}{p^+} + \frac{g^2}{48\pi^2} \int d\tau d\sigma \left( \frac{\partial q^{\wedge}}{\partial \sigma} \right)^2 \quad (130)$$

When exponentiated (through higher loop corrections) this expression can be interpreted as adding new terms to  $-S$ , where  $S$  is the worldsheet action. The first term modifies the treatment of the worldsheet boundary in a similar way to the description of a mass term [5], and the second term is a new bulk term. Both terms violate helicity in the right way to cancel the helicity violation implied by the nonvanishing of  $\Pi^{\wedge\wedge}$ . These new terms also produce new effects from contact contributions arising when the bulk term sits

on the same time slice as an interaction vertex, similarly to the generation of quartic vertices from two cubics [3]. Using the generating function obtained in that reference we find, for correlators on the same time slice,

$$\left\langle \left( \frac{q_{i+1} - q_i}{m} \right)^\vee \left( \frac{q_{j+1} - q_j}{m} \right)^{\wedge 2} \right\rangle = \frac{p^{\wedge 2} p^\vee}{p^{+3}} + \frac{2}{a} \frac{p^\vee}{p^{+2}} \left[ \frac{\delta_{ij}}{m} - \frac{1}{p^+} \right] \quad (131)$$

$$am \sum_j \left\langle \left( \frac{q_{i+1} - q_i}{m} \right)^\vee \left( \frac{q_{j+1} - q_j}{m} \right)^{\wedge 2} \right\rangle = a \frac{p^{\wedge 2} p^\vee}{p^{+2}} \quad (132)$$

$$\begin{aligned} \left\langle \left( \frac{q_{i+1} - q_i}{m} \right)^\vee \left( \frac{q_{j+1} - q_j}{m} \right)^{\wedge 2} \left( \frac{q_{l+1} - q_l}{m} \right)^\vee \right\rangle &= \frac{p^{\wedge 2} p^{\vee 2}}{p^{+4}} + \frac{2}{a} \frac{p^\wedge p^\vee}{p^{+2}} \left[ \frac{\delta_{ij} + \delta_{lj}}{m} - \frac{2}{p^+} \right] \\ &+ \frac{2}{a^2} \left[ \frac{\delta_{ij}}{m} - \frac{1}{p^+} \right] \left[ \frac{\delta_{lj}}{m} - \frac{1}{p^+} \right] \end{aligned} \quad (133)$$

$$am \sum_j \left\langle \left( \frac{q_{i+1} - q_i}{m} \right)^\vee \left( \frac{q_{j+1} - q_j}{m} \right)^{\wedge 2} \left( \frac{q_{l+1} - q_l}{m} \right)^\vee \right\rangle = a \frac{p^{\wedge 2} p^{\vee 2}}{p^{+3}} + \frac{2}{a} \left[ \frac{\delta_{il}}{m} - \frac{1}{p^+} \right] \quad (134)$$

Here  $a, m$  are the discretization units of  $\tau, \sigma$  respectively. The terms proportional to  $a$  will be negligible in the continuum limit, so we see that the only contact contribution that survives is the last term on the last line. Its effect is to produce new quartic vertices similar to the quartic anomalies already discussed. Indeed the usual quartic vertices arise from the second term of the correlator

$$\left\langle \left( \frac{q_{i+1} - q_i}{m} \right)^\vee \left( \frac{q_{j+1} - q_j}{m} \right)^\wedge \right\rangle = \frac{p^\wedge p^\vee}{p^{+2}} + \frac{1}{a} \left[ \frac{\delta_{ij}}{m} - \frac{1}{p^+} \right] \quad (135)$$

which has an exactly analogous  $1/a$  contribution. By a parallel calculation we easily find that the contraction terms produced by  $\Pi_{\text{C.T.}}^{\wedge\wedge}$  give the following quartic vertices:

$$\Gamma_{\Pi}^{\wedge\wedge\vee\vee} = -\frac{g^4}{12\pi^2} \left[ \frac{p_1^+ p_3^+ + p_2^+ p_4^+}{p_{14}^{+2}} + 1 \right] \quad (136)$$

$$\Gamma_{\Pi}^{\wedge\vee\wedge\vee} = \frac{g^4}{12\pi^2} \left[ \frac{p_1^+ p_2^+ + p_3^+ p_4^+}{p_{14}^{+2}} + \frac{p_1^+ p_4^+ + p_2^+ p_3^+}{p_{12}^{+2}} + 2 \right] \quad (137)$$

These new terms add to the anomalous contributions previously discussed:

$$\Gamma_{\text{anom}}^{\wedge\wedge\vee\vee} \rightarrow \Gamma_{\text{anom ws}}^{\wedge\wedge\vee\vee} = \frac{g^2}{24\pi^2} \left[ -4g^2 \frac{p_1^+ p_3^+ + p_2^+ p_4^+}{(p_1^+ + p_4^+)^2} \right] \quad (138)$$

$$\Gamma_{\text{anom}}^{\wedge\vee\wedge\vee} \rightarrow \Gamma_{\text{anom ws}}^{\wedge\vee\wedge\vee} = \frac{g^2}{24\pi^2} \left[ 4g^2 \frac{p_1^+ p_2^+ + p_3^+ p_4^+}{(p_1^+ + p_4^+)^2} + 4g^2 \frac{p_1^+ p_4^+ + p_2^+ p_3^+}{(p_1^+ + p_2^+)^2} + 6g^2 \right] \quad (139)$$

The nonlocal terms in these expressions can be handled as before by retuning the  $p^2$  term in  $\Pi^{\wedge\vee}$  a little differently:

$$\Pi_{\text{C.T.}}^{\wedge\vee} \rightarrow \Pi_{\text{C.T. ws}}^{\wedge\vee} = -\frac{g^2}{24m\delta} p^+ + \frac{g^2}{4\pi^2\delta} + \frac{g^2}{12\pi^2} p^2 \quad (140)$$

The worldsheet description of the first two terms in  $\Pi_{\text{C.T. ws}}^{\wedge\vee}$  has been explained in [4]: the first term can be absorbed in a worldsheet boundary cosmological constant, and the second term, which has the interpretation as a shift in the gluon (mass)<sup>2</sup>, can be absorbed in the worldsheet description of a mass counterterm [5]. The way to put the last term in the worldsheet description is a little more subtle. We first note that its effect is simply a finite contribution to wave function renormalization  $Z$  which multiplies the cubic vertex by  $Z^{3/2}$  and the quartic vertex by  $Z^2$ . It is convenient to, at the same time, make a finite renormalization

of the coupling constant  $g \rightarrow g/Z$  to reduce the effect to multiplying the cubic vertex by  $Z^{1/2}$  leaving the quartic vertex untouched. Then the net effect, to be described by the worldsheet formalism, is to modify the constant part of the correction multiplying the part of the tree involving gluon exchange from  $73/9 \rightarrow 67/9$  without touching the correction multiplying the quartic vertex part of the tree, which gets adjusted to  $67/9$  by the nonlocal part of the quartic anomaly. One's first thought is to simply change the coefficient of the cubic vertex appropriately. But then the worldsheet contact contributions would make a corresponding modification to the quartic vertex contribution and the change would only amount to a finite renormalization of  $g$ . Fortunately it is possible to prevent the  $\partial q/\partial\sigma$  worldsheet insertions from generating quartic contributions by altering the ghost worldsheet action near the interaction point.

The crucial feature of the ghost path integral that permits this was explained in [2]. The discretized ( $p^+ = Mm$ ) ghost action on a fixed time slice is

$$\sum_{i=0}^{M-1} (b_{i+1} - b_i)(c_{i+1} - c_i) \quad (141)$$

where  $b_0, b_M, c_0, c_M = 0$ . The effect of the worldsheet integral of the exponential of this expression is to supply a factor of  $M$  which cancels a  $1/M$  from the coordinate part of the path integral. If a single link in the sum is deleted, the result of integration is down by  $1/M$  in other words it is 1. This was the mechanism we used to generate needed  $1/p^+$  factors in the vertex functions. The location of these ghost deletions is indicated by short vertical lines in Fig. 13. If *two* links are deleted on the same time slice of the same gluon

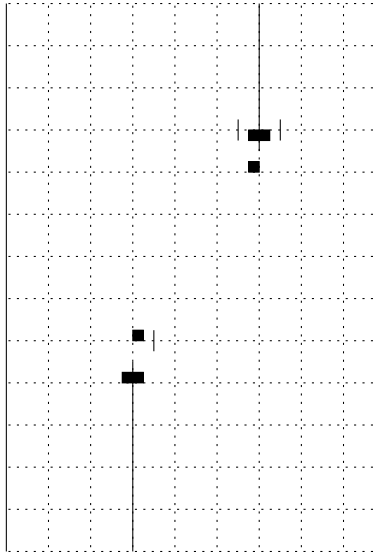


Figure 13: Discretized worldsheet for a four gluon tree. The solid squares indicate where  $\partial q/\partial\sigma$  insertions can be located. The short vertical lines indicate the links to be deleted in the worldsheet ghost action. All indicated ghost link deletions are present regardless of the insertion location.

propagator, the worldsheet integral gives zero. Thus, when the insertions are on the same time slice there will be *two* deleted links and the contribution will be suppressed. Since the deleted links produce unwanted  $1/p^+$  factors, one must also include dummy ghost insertions (defined in [3]) to put back corresponding factors of  $p^+$ . Thus we can suppress contact contributions from being produced by the cubic counterterms by accompanying each  $\partial q/\partial\sigma$  insertion with an extra deleted link. We show in Fig. 14 which ghost link deletion is made for each of the six possible  $\partial q/\partial\sigma$  insertions (three for the fusion vertex and three for the fission vertex). This ghost deletion scheme allows us in effect to change the cubic vertex by a multiple of

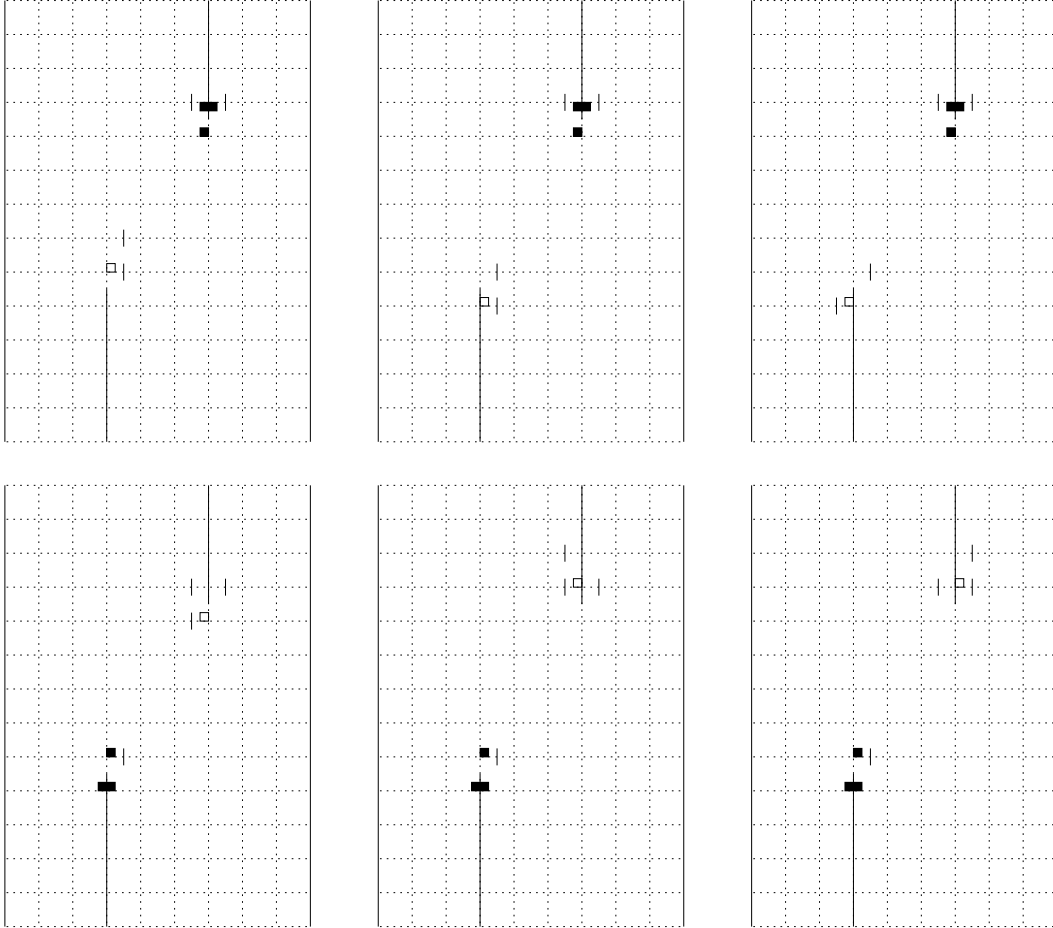


Figure 14: Discretized worldsheet for a four gluon amplitude with one tree cubic vertex (solid squares) and one cubic counterterm vertex (open squares). The squares indicate where  $\partial q/\partial\sigma$  insertions can be located. The short vertical lines indicate the links to be deleted in the worldsheet ghost action. Notice that each counterterm insertion is accompanied by an extra ghost link deletion. Inspection shows that whenever two insertions are on the same time slice of the same gluon propagator, there are *two* deletions and hence the contact contribution is suppressed.



itself without affecting the quartic vertex, and this in turn allows the conversion of the nonlocal parts of the quartic counterterms to complete trees.

The same scheme is very useful in translating the cubic counterterm (125) to the worldsheet.

$$\begin{aligned}\Gamma_{\text{C.T.}}^{\wedge\wedge\vee} &= \frac{g^3}{12\pi^2}[k_0^\wedge + k_1^\wedge + k_2^\wedge] = \frac{g^3}{12\pi^2}[k_0^\wedge - k_1^\wedge + k_2^\wedge - k_1^\wedge + 3k_1^\wedge] \\ &= \frac{g^3}{12\pi^2}[p_2^\wedge - p_1^\wedge + 3k_1^\wedge] = \frac{g^3}{12\pi^2}\left\langle p_2^+ \frac{\partial q^\wedge}{\partial \sigma}(B) - p_1^+ \frac{\partial q^\wedge}{\partial \sigma}(A) + 3q^\wedge(A) \right\rangle\end{aligned}\quad (142)$$

Here  $A$  and  $B$  label worldsheet points just to the left and right of the internal boundary separating the two gluon propagators that fuse to or fission from the third gluon propagator (see Fig. 15). We can then use

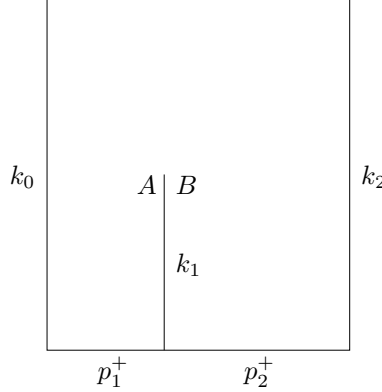


Figure 15: Worldsheet for cubic fusion vertex.

the ghost deletion scheme just described to guarantee that these insertions produce no modification of the quartic counterterms.

We have now shown how the self-energy and the cubic counterterms, together with the nonlocal parts (in target space) of the quartic counterterms can all be given a local worldsheet description. It remains to find a local worldsheet description of the purely constant parts of the quartic counterterms

$$\Gamma_{\text{C.T. ws}}^{\wedge\wedge\vee\vee} = 0, \quad \Gamma_{\text{C.T. ws}}^{\wedge\vee\wedge\vee} = -\frac{g^4}{4\pi^2}\quad (143)$$

We cannot simply postulate a direct quartic interaction vertex in the worldsheet formalism without destroying worldsheet locality. We therefore search for a cubic vertex whose contact contributions generate constant quartic vertices. Consider the simple ansatz

$$C^{\wedge\wedge\vee} = g^3\xi(p_2^\wedge - p_1^\wedge) \rightarrow g^3\xi\left\langle p_2^+ \frac{\partial q^\wedge}{\partial \sigma}(2) - p_1^+ \frac{\partial q^\wedge}{\partial \sigma}(1) \right\rangle\quad (144)$$

where the legs of the vertex are labeled 1, 2, 3 counterclockwise and 1, 2 have like helicity. Then it is not hard to see that the four gluon trees built from one tree cubic and one of these vertices, generate the contact contributions

$$C^{\wedge\wedge\vee\vee} = -g^4\xi, \quad C^{\wedge\vee\wedge\vee} = +2g^4\xi\quad (145)$$

That is the ratio of the two polarization structures is the same as that coming from the  $[A_\mu, A_\nu]^2$  term, in the field theoretic Lagrangian. To make this work we need to suppress the new cubic couplings while retaining the contact contributions. To do this we can write

$$0 = C^{\wedge\wedge\vee} - C^{\wedge\wedge\vee}\quad (146)$$

and apply ghost link deletions on the second term so that it will not produce contact contributions. Then the exchange graphs will cancel leaving only the contact quartic vertex!

We have found no simple variation of this scheme that provides us with exactly the counterterm (143). Instead, our proposal is to increase the flexibility of the worldsheet formalism by increasing the dimensionality of the worldsheet fields  $\mathbf{q}(\sigma, \tau)$ . This is not unprecedented. Recall that in the *AdS/CFT* correspondence [16], the string theory is formulated in ten space-time dimensions whereas the supersymmetric gauge theory is formulated in only four space-time dimensions. Similarly, in developing the worldsheet description of  $\mathcal{N} = 4$  supersymmetric gauge theories [17], we found it particularly convenient to add six extra dimensions, that is the index of  $q^i$  took the values  $i = 1, 2, \dots, 8$ . The boundary conditions on the six new  $q$ 's were strict Dirichlet conditions  $q^i = 0$  on all boundaries, internal or external. At the same time we added three new sets of  $b, c$  ghosts, which like the original set have strict Dirichlet boundary conditions. Since the extra  $q$ 's and ghosts share identical boundary conditions, their contributions to the path integral exactly cancel: they are just dummy integration variables.

For our purposes, to locally produce the necessary quartic counterterms in pure gauge theories, two extra dimensions and one extra set of  $b, c$  ghosts suffice. We thus have four transverse dual momenta corresponding to six dimensional space-time. Let us call the new dimensions  $r^k$ ,  $k = 1, 2$  and we can, if we wish, use a complex basis  $r^\wedge, r^\vee$ . But here  $\wedge, \vee$  do *not* represent helicity, but rather an analogous charge in the extra dimensions. Next we allow spurions with values  $\pm 1$  of this charge to couple to two gluons as indicated in the top line Fig. 16. In order to guarantee that the spurion decouples, we insert a factor  $p_r^+ \partial r / \partial \sigma$  on the

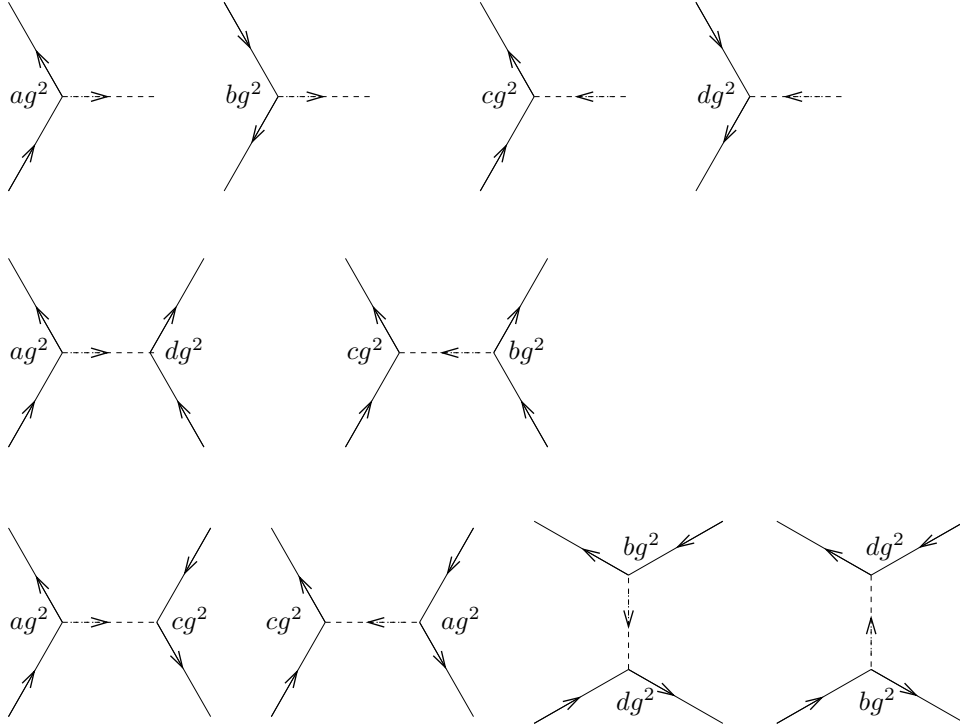


Figure 16: New cubic counterterms, with a spurion (dashed line) coupling to two gluons.

spurion propagator near the interaction point. Because  $r^k = 0$  on all worldsheet boundaries, the average of this factor over worldsheet fields vanishes, except when there is another such factor on the same time slice. In other words all the four gluon diagrams exchanging the spurion with internal propagators vanish, leaving only the contact contributions. By inspecting the coupling assignments shown in Fig. 16, we see that

$$\Gamma_{spur}^{\wedge\wedge\vee\vee} = -(ad + bc)g^4 \quad (147)$$

$$\Gamma_{spur}^{\wedge\wedge\wedge\vee} = -2(ac + bd)g^4 \quad (148)$$

Altogether we have five adjustable parameters to produce two independent counterterms:

$$\Gamma_{C.T.}^{\wedge\wedge\wedge\vee} = C^{\wedge\wedge\wedge\vee} + \Gamma_{spur}^{\wedge\wedge\wedge\vee} = -(ad + bc + \xi)g^4 \quad (149)$$

$$\Gamma_{C.T.}^{\wedge\vee\wedge\vee} = C^{\wedge\vee\wedge\vee} + \Gamma_{spur}^{\wedge\vee\wedge\vee} = -2(ac + bd - \xi)g^4 \quad (150)$$

Since  $\xi, a, b, c, d$  are arbitrary, there is more than enough flexibility to produce the necessary counterterms, and more generally to adjust them appropriately at each order in perturbation theory. It is perhaps most economical to employ the extra dimensions only to cancel the part of the anomaly due to UV artifacts, which would demand spin independence for this part:  $ad + bc = 2(ac + bd)$ . With this choice we then determine  $\xi = -(ad + bc) = -1/12\pi^2$ .

## 11 Conclusion

In this paper we have completed the lightcone gauge calculations of the scattering of glue by glue through one loop. Our results completely agree with those obtained using covariant methods [20]. In addition to obtaining the elastic amplitudes through one loop which are divergent in the infrared, we have also calculated their contributions to probabilities and have shown that infrared divergences cancel against contributions from extra gluons in the initial and final states. This is all in accord with the Lee-Nauenberg theorem.

The expressions for the final infrared finite probabilities, including the bremsstrahlung gluons, are extremely compact, and are manifestly Lorentz invariant. Infrared divergences have been traded for a resolution parameter  $\Delta$  characterizing unseen gluons and jets.

All calculations were done in 4 space-time dimensions without the benefit of dimensional regularization, which means that (local) counterterms beyond wave function and coupling renormalization must be included. In spite of artificial  $p^+ = 0$  divergences (which raise the ugly possibility of requiring nonpolynomial counterterms), all necessary counterterms are polynomials in the external momenta of the degree dictated by power counting. We would like to underline here the fact that we did nothing sophisticated with  $p^+$  zero modes. We simply discretized the  $p^+$  integrals and omitted the zero modes [21, 22]. Apart from collinear divergences, which are only a problem for self energy insertions on external lines, this discretization provides an apt infrared regulator for lightcone calculations. Our calculation shows that in all infrared safe calculations, including on-shell gluon scattering with due care taken with jets and gluon bremsstrahlung, the continuum limit of this discretization is finite and yields all previously known results obtained in covariant gauges [18–20]. *A sophisticated treatment of  $p^+$  zero modes is not required.*

Finally we have discussed how to incorporate all of the counterterms we require in the lightcone worldsheet formalism. A particularly convenient way to do this is to interpret the QCD “string” dynamics as occurring in 6 dimensional space-time. We stress that the extra two dimensions are holographically generated on the “string” side of Field/String duality and are not present at all in the field theoretic description of the “field” side of the duality. Significantly, this can all be done while preserving a local worldsheet dynamics.

For perturbative QCD, the next step is to prove that the lightcone gauge calculational procedure we have adopted carries through to all orders in perturbation theory. We believe it will because we have only needed to introduce strictly local counterterms, consistent with the concept that gauge violating artifacts are entirely associated with the ultraviolet part of the dynamics. The fact that we needed some counterterms that were not in the input classical Lagrangian is completely standard with the use of a gauge noninvariant regulation and should not obstruct the usual renormalization program. This is because the new counterterms introduced obey the power counting rules of renormalizable theories.

If the renormalization program goes through as we expect, then the modifications we have made in the “bare” worldsheet description to accommodate the counterterms should suffice to all orders in perturbation theory. This would fulfill our ambitious goal of establishing a string theory dual of the large  $N_c$  limit of QCD, working entirely from the field theory side of the duality. Then the exciting prospect before us would be to use this duality to deepen our insight into nonperturbative aspects of the strong interactions.

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## A Bremsstrahlung Integrals

In the evaluation of soft and collinear bremsstrahlung cross sections we need to do several integrations over phase space. For the hard collinear case the integral was simple enough to treat in the text. Here we sketch the evaluation of the integrals for soft radiation which are more complex.

At  $N_c = \infty$  each amplitude is the sum of gluon emissions from two neighboring lines, so the squared amplitude has two direct terms and a cross term. We first consider the transverse integration over the resolution of a direct term, which has the structure

$$\begin{aligned} I_{\text{Direct}}(\mathbf{v}) &= \frac{1}{2} \int d\mathbf{k} \frac{(\mathbf{k} + \mathbf{v})^2}{[(\mathbf{k} + \mathbf{v})^2 + M^2]^2} \\ &= \frac{1}{2} \int_0^{\Delta'} k dk \int_0^{2\pi} d\phi \frac{k^2 + v^2 + 2kv \cos \phi}{(k^2 + v^2 + M^2 + 2kv \cos \phi)^2} \end{aligned} \quad (151)$$

The  $\phi$  integral is easily done by transforming to a contour integral over  $z = e^{i\phi}$  and evaluating residues. Changing variables from  $k$  to  $t = k^2$  then leaves us with

$$I_{\text{Direct}}(\mathbf{v}) = \frac{\pi}{2} \int_0^{\Delta'^2} dt \frac{(t - v^2)^2 + M^2(t + v^2)}{[(t - v^2)^2 + 2M^2(t + v^2) + M^4]^{3/2}} \quad (152)$$

$$\begin{aligned} &= -\frac{\pi}{4} \frac{(\Delta'^2 - v^2) - M^2}{\sqrt{(\Delta'^2 - v^2)^2 + M^4 + 2\Delta'^2 M^2 + 2M^2 v^2}} - \frac{\pi}{4} \\ &\quad + \frac{\pi}{2} \ln \frac{(M^2 - v^2 + \Delta'^2 + \sqrt{(\Delta'^2 - v^2)^2 + M^4 + 2\Delta'^2 M^2 + 2M^2 v^2})}{2M^2} \end{aligned} \quad (153)$$

$$\sim \frac{\pi}{2} \ln \frac{\Delta'^2 - v^2}{M^2 e} \quad \text{for } \Delta'^2 > v^2; \quad \frac{\pi}{2} \ln \frac{v^2}{v^2 - \Delta'^2} \quad \text{for } \Delta'^2 < v^2 \quad (154)$$

where the last line applies as  $M \rightarrow 0$ .

The transverse momentum integral of a cross term has the structure

$$\begin{aligned} I_{\text{Cross}} &= \int d\mathbf{k} \frac{(\mathbf{k} + \mathbf{v}) \cdot (\mathbf{k} + \mathbf{w})}{[(\mathbf{k} + \mathbf{v})^2 + M^2][(\mathbf{k} + \mathbf{w})^2 + M^2]} \\ &= \int_0^{\Delta'} k dk \int_0^{2\pi} d\phi \frac{k^2 + v w \cos \alpha + k v \cos \phi + k w \cos(\phi - \alpha)}{(k^2 + v^2 + M^2 + 2kv \cos \phi)(k^2 + w^2 + M^2 + 2kw \cos(\phi - \alpha))} \end{aligned} \quad (155)$$

Again the  $\phi$  integral is easily done by converting to a contour integral in  $z = e^{i\phi}$ . In this case we can simplify life by taking  $M = 0$  from the beginning, in which case the poles of the integrand are at  $z = -v/k, -k/v, -we^{i\alpha}/k, -ke^{i\alpha}/w$ . The contour at the unit circle will enclose precisely two of these poles:  $z = -\min(v/k, k/v), -e^{i\alpha} \min(w/k, k/w)$ , with four possibilities depending on the relative size of  $k, v, w$ . For definiteness let's assume that  $v > w$ . Then when  $w < k < v$  it turns out that the two pole contributions exactly cancel. Then the angular integral is

$$\begin{aligned} &\pi \left[ \frac{1}{k^2 - v w e^{-i\alpha}} + \frac{1}{k^2 - v w e^{i\alpha}} \right], \quad \text{for } k > v > w \\ &-\pi \left[ \frac{1}{k^2 - v w e^{-i\alpha}} + \frac{1}{k^2 - v w e^{i\alpha}} \right], \quad \text{for } v > w > k \\ &0, \quad \text{for } v > k > w \end{aligned} \quad (156)$$

and we get with  $u = k^2$ , assuming  $\Delta'^2 > v^2, w^2$ ,

$$\begin{aligned}
I_{\text{Cross}} &= -\frac{\pi}{2} \int_0^{w^2} du \left[ \frac{1}{u - vwe^{-i\alpha}} + \frac{1}{u - vwe^{i\alpha}} \right] + \frac{\pi}{2} \int_{v^2}^{\Delta'^2} du \left[ \frac{1}{u - vwe^{-i\alpha}} + \frac{1}{u - vwe^{i\alpha}} \right] \\
&= \frac{\pi}{2} \ln \frac{\Delta'^4 - 2\Delta'^2 vw \cos \alpha + v^2 w^2}{(v^2 + w^2 - 2vw \cos \alpha)^2} \\
&= \frac{\pi}{2} \ln \frac{\Delta'^4 - 2\Delta'^2 \mathbf{v} \cdot \mathbf{w} + v^2 w^2}{(\mathbf{v} - \mathbf{w})^4}, \quad \text{for } w^2 < v^2 < \Delta'^2; \\
&\quad -\frac{\pi}{2} \ln \frac{(\mathbf{v} - \mathbf{w})^2}{v^2}, \quad \text{for } w^2 < \Delta'^2 < v^2; \\
&\quad -\frac{\pi}{2} \ln \frac{\Delta'^4 - 2\Delta'^2 \mathbf{v} \cdot \mathbf{w} + v^2 w^2}{v^2 w^2}, \quad \text{for } \Delta'^2 < w^2 < v^2
\end{aligned}$$

If  $v < w$ , the same result follows. The complete transverse integral for soft gluon radiation is the combination

$$\begin{aligned}
I_{\text{Total}} &= I_{\text{Direct}}(\mathbf{v}) + I_{\text{Direct}}(\mathbf{w}) - I_{\text{Cross}} \\
&= \frac{\pi}{2} \ln \frac{(\Delta'^2 - v^2)(\Delta'^2 - w^2)}{\Delta'^4 - 2\Delta'^2 \mathbf{v} \cdot \mathbf{w} + v^2 w^2} + \pi \ln \frac{(\mathbf{v} - \mathbf{w})^2}{M^2 e}, \quad \text{for } w^2 < v^2 < \Delta'^2; \\
&= \frac{\pi}{2} \ln \frac{(\Delta'^2 - w^2)}{v^2 - \Delta'^2} + \frac{\pi}{2} \ln \frac{(\mathbf{v} - \mathbf{w})^2}{M^2 e}, \quad \text{for } w^2 < \Delta'^2 < v^2; \\
&= -\frac{\pi}{2} \ln \frac{(\Delta'^2 - v^2)(\Delta'^2 - w^2)}{\Delta'^4 - 2\Delta'^2 \mathbf{v} \cdot \mathbf{w} + v^2 w^2}, \quad \text{for } \Delta'^2 < w^2 < v^2
\end{aligned}$$

For  $k^+ \rightarrow 0$  both  $v/\Delta', w/\Delta' \rightarrow 0$  so we see that the small  $k^+$  region is insensitive to the resolution  $\Delta'$ .

## B Evaluation of Bubble and Triangle Integrals

We list here the  $q^-$  integrations of bubble and triangle integrands that are useful in analyzing collinear divergences. First the six bubble integrands:

$$-i \int \frac{dq^-}{2\pi} \frac{1}{q_0^2 q_1^2} = \frac{1}{2p_1^+ (q_0 - q_0^+ \mathbf{p}_1/p_1^+)^2}, \quad \text{for } 0 < q_0^+ < p_1^+ \quad (157)$$

$$-i \int \frac{dq^-}{2\pi} \frac{1}{q_1^2 q_2^2} = \frac{1}{2p_2^+ (q_1 - q_1^+ \mathbf{p}_2/p_2^+)^2}, \quad \text{for } 0 < q_1^+ < p_2^+ \quad (158)$$

$$-i \int \frac{dq^-}{2\pi} \frac{1}{q_2^2 q_3^2} = \frac{-1}{2p_3^+ (q_3 - q_3^+ \mathbf{p}_3/p_3^+)^2}, \quad \text{for } 0 < q_3^+ < -p_3^+ \quad (159)$$

$$-i \int \frac{dq^-}{2\pi} \frac{1}{q_0^2 q_3^2} = \frac{-1}{2p_4^+ (q_0 - q_0^+ \mathbf{p}_4/p_4^+)^2}, \quad \text{for } 0 < q^+ < -p_4^+ \quad (160)$$

$$-i \int \frac{dq^-}{2\pi} \frac{1}{q_0^2 q_2^2} = \frac{1}{2p_{12}^+ [(q_0 - q_0^+ \mathbf{p}_{12}/p_{12}^+)^2 + q_0^+ (p_{12}^+ - q_0^+) p_{12}^2/p_{12}^{+2}]}, \quad \text{for } 0 < q_0^+ < p_{12}^+ \quad (161)$$

$$-i \int \frac{dq^-}{2\pi} \frac{1}{q_1^2 q_3^2} = \frac{1}{2|p_{14}^+| [(q_1 - q_1^+ \mathbf{p}_{14}/p_{14}^+)^2 - q_1^+ (q_1^+ + p_{14}^+) p_{14}^2/p_{14}^{+2}]}, \quad \text{for } 0 < |q_1^+| < |p_{14}^+| \quad (162)$$

Here we recall the notation  $q_i \equiv q - k_i$ , where  $k_i$  are the dual momenta, related to the gluon momenta by  $p_i = k_i - k_{i-1}$ , with  $k_4 \equiv k_0$ . We normally take  $k_0^\pm = 0$ . Next we list the triangle integrals:

$$-i \int \frac{dq^-}{2\pi} \frac{1}{q_0^2 q_1^2 q_2^2}$$

$$\begin{aligned}
&= \frac{q_0^+}{2p_1^+p_{12}^+(\mathbf{q}_0 - q_0^+\mathbf{p}_1/p_1^+)^2[(\mathbf{q}_0 - q_0^+\mathbf{p}_{12}/p_{12}^+)^2 + q_0^+(p_{12}^+ - q_0^+)p_{12}^2/p_{12}^{+2}]}, \quad \text{for } 0 < q_0^+ < p_1^+ \\
&= \frac{p_{12}^+ - q_0^+}{2p_2^+p_{12}^+(\mathbf{q}_0 - \mathbf{p}_{12} - (q_0^+ - p_{12}^+)\mathbf{p}_2/p_2^+)^2[(\mathbf{q}_0 - q_0^+\mathbf{p}_{12}/p_{12}^+)^2 + q_0^+(p_{12}^+ - q_0^+)p_{12}^2/p_{12}^{+2}]}, \quad \text{for } p_1^+ < q_0^+ < p_{12}^+ \\
&-i \int \frac{dq^-}{2\pi} \frac{1}{q_0^2 q_1^2 q_3^2} \\
&= \frac{-q_0^+}{2p_1^+p_4^+(\mathbf{q}_0 - q_0^+\mathbf{p}_1/p_1^+)^2(\mathbf{q}_0 - q_0^+\mathbf{p}_4/p_4^+)^2}, \quad \text{for } 0 < q_0^+ < p_1^+ \\
&= \frac{-q_0^+ - p_4^+}{2p_4^+p_{14}^+(\mathbf{q}_0 - q_0^+\mathbf{p}_4/p_4^+)^2[(\mathbf{q}_0 + \mathbf{p}_4 - (q_0^+ + p_4^+)\mathbf{p}_{14}/p_{14}^+)^2 - (q_0^+ + p_4^+)(q_0^+ - p_1^+)p_{14}^2/p_{14}^{+2}]}, \\
&\quad \text{for } p_1^+ < q_0^+ < -p_4^+ \\
&-i \int \frac{dq^-}{2\pi} \frac{1}{q_1^2 q_2^2 q_3^2} \\
&= \frac{-q_1^+}{2p_2^+p_{14}^+(\mathbf{q}_1 - q_1^+\mathbf{p}_2/p_2^+)^2[(\mathbf{q}_1 - q_1^+\mathbf{p}_{14}/p_{14}^+)^2 - (q_1^+ + p_{14}^+)q_1^+p_{14}^2/p_{14}^{+2}]}, \\
&\quad \text{for } 0 < q_1^+ < -p_{14}^+ \\
&= \frac{q_1^+ - p_2^+}{2p_2^+p_3^+(\mathbf{q}_1 - q_1^+\mathbf{p}_2/p_2^+)^2(\mathbf{q}_1 - \mathbf{p}_2 - (q_1^+ - p_2^+)\mathbf{p}_3/p_3^+)^2}, \quad \text{for } -p_{14}^+ < q_1^+ < p_2^+ \\
&-i \int \frac{dq^-}{2\pi} \frac{1}{q_0^2 q_2^2 q_3^2} \\
&= \frac{-q_0^+}{2p_{12}^+p_4^+[(\mathbf{q}_0 - q^+\mathbf{p}_{12}/p_{12}^+)^2 + q_0^+(p_{12}^+ - q_0^+)p_{12}^2/p_{12}^{+2}](\mathbf{q}_0 - q^+\mathbf{p}_4/p_4^+)^2}, \quad \text{for } 0 < q_0^+ < -p_4^+ \\
&= \frac{q_0^+ - p_{12}^+}{2p_3^+p_{12}^+(\mathbf{q}_0 - \mathbf{p}_{12} - (q_0^+ - p_{12}^+)\mathbf{p}_3/p_3^+)^2[(\mathbf{q}_0 - q^+\mathbf{p}_{12}/p_{12}^+)^2 + q_0^+(p_{12}^+ - q_0^+)p_{12}^2/p_{12}^{+2}]}, \\
&\quad \text{for } -p_4^+ < q_0^+ < p_{12}^+
\end{aligned}$$

Once the  $q^-$  integrals have been performed, the transverse  $\mathbf{q}$  integrals can be done using one or two Schwinger parameters to exponentiate the one or two denominators.

## C Evaluation of Box Integrals

The box integrals we encounter can be most easily handled through the introduction of Schwinger parameters  $T_1, T_2, T_3, T_4$  for the internal line propagators  $(q - k_0)^{-2}, (q - k_1)^{-2}, (q - k_2)^{-2}, (q - k_3)^{-2}$  respectively. Since some of them are divergent in the ultra-violet, we also retain the worldsheet UV cutoff factors  $e^{-\delta \mathbf{q}^2}$ . The integration over  $q$  is then a Gaussian that is easily done by completing the square and shifting  $q \rightarrow q + K$ , with

$$\mathbf{K} = \frac{\mathbf{k}_0 T_1 + \mathbf{k}_1 T_2 + \mathbf{k}_2 T_3 + \mathbf{k}_3 T_4}{T_{14} + \delta}, \quad K^\pm = \frac{(k_0 T_1 + k_1 T_2 + k_2 T_3 + k_3 T_4)^\pm}{T_{14}} \quad (163)$$

where we use the shorthand  $T_{14} = T_1 + T_2 + T_3 + T_4$ . One then finds, using the Feynman parameters  $x_i \equiv T_i/T_{14}$  that

$$\mathbf{K}_{16} \rightarrow -p_1^+ \mathbf{q} + q^+ \mathbf{p}_1 - x_3 \mathbf{K}_{12} - x_4 \mathbf{K}_{41} + p_1^+ \frac{\delta \mathbf{K}}{T_{14}} \quad (164)$$

$$\mathbf{K}_{52} \rightarrow -p_2^+ \mathbf{q} + q^+ \mathbf{p}_2 - x_4 \mathbf{K}_{23} - x_1 \mathbf{K}_{12} + p_2^+ \frac{\delta \mathbf{K}}{T_{14}} \quad (165)$$

$$\mathbf{K}_{35} \rightarrow p_3^+ \mathbf{q} - q^+ \mathbf{p}_3 + x_2 \mathbf{K}_{23} + x_1 \mathbf{K}_{34} - p_3^+ \frac{\delta \mathbf{K}}{T_{14}} \quad (166)$$

$$\mathbf{K}_{64} \rightarrow p_4^+ \mathbf{q} - q^+ \mathbf{p}_4 + x_3 \mathbf{K}_{34} + x_2 \mathbf{K}_{41} - p_4^+ \frac{\delta \mathbf{K}}{T_{14}} \quad (167)$$

and the Gaussian factor left over is just  $\exp\{-T_{14}(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2) + O(\delta)\}$ . The  $O(\delta)$  term in the exponent is negligible in the box integrals because the log divergence is insufficient to overwhelm it. Also when the  $\mathbf{K}_{ij}$  occur in the numerator of the box integrand the terms  $p_i^+ \delta \mathbf{K}/T_{14}$  are negligible since they are  $O(1)$  only when all  $T_i = O(\delta)$  and they occur only in integrals convergent in this region. The Gaussian integration over  $q$  involves up to four powers of  $q$  as prefactors.

$$\begin{aligned} \int d^4 q e^{-T_{14} q^2 - \delta q^2} &= \frac{\pi^2}{T_{14}(T_{14} + \delta)}, & \int d^4 q q^\wedge q^\vee e^{-T_{14} q^2 - \delta q^2} &= \frac{\pi^2}{2T_{14}(T_{14} + \delta)^2} \\ \int d^4 q q^2 e^{-T_{14} q^2 - \delta q^2} &= \frac{\pi^2(2T_{14} + \delta)}{T_{14}^2(T_{14} + \delta)^2}, & \int d^4 q (q^\wedge q^\vee)^2 e^{-T_{14} q^2 - \delta q^2} &= \frac{\pi^2}{2T_{14}(T_{14} + \delta)^3} \\ \int d^4 q q^\wedge q^\vee q^2 e^{-T_{14} q^2 - \delta q^2} &= \frac{\pi^2(3T_{14} + \delta)}{2T_{14}^2(T_{14} + \delta)^3} \end{aligned} \quad (168)$$

Changing variables from the  $T_i$  to three of the  $x_i$  and  $T_{14} = T$  reduces the integration measure to  $d^4 x \delta(1 - \sum x_i) T^3 dT$ . In the first three cases it is safe to set  $\delta = 0$ , and the integral over  $T$  gives

$$\frac{\pi^2}{(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2)^2}, \quad \frac{\pi^2}{2(x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2)}, \quad \frac{2\pi^2}{x_1 x_3 p_{12}^2 + x_2 x_4 p_{14}^2} \quad (169)$$

respectively. In the last two cases we must evaluate integrals that are log divergent for  $\delta \rightarrow 0$ . We find, as  $\delta \rightarrow 0$ ,

$$\begin{aligned} \int_0^\infty \frac{T^n dT}{(T + \delta)^{n+1}} e^{-TH} &\sim -\sum_{k=1}^n \frac{1}{k} - \gamma - \ln(\delta H) \\ \int_0^\infty \frac{T^2 dT}{(T + \delta)^3} e^{-TH} &\sim -\frac{3}{2} - \gamma - \ln(\delta H), & \int_0^\infty \frac{T(3T + \delta) dT}{(T + \delta)^3} e^{-TH} &\sim -4 - 3\gamma - 3 \ln(\delta H) \end{aligned} \quad (170)$$

where  $\gamma = -\Gamma'(1)$  is Euler's constant.

We shall have use for the following combinations of momenta which arise in the box integrand after shifting  $q$  and sending  $\delta \rightarrow 0$

$$K_0 = x_2 p_1 + x_3(p_1 + p_2) - x_4 p_4 \quad (171)$$

$$K_0 - p_1 = x_3 p_2 + x_4(p_2 + p_3) - x_1 p_1 \quad (172)$$

$$K_0 - p_1 - p_2 = x_4 p_3 + x_1(p_3 + p_4) - x_2 p_2 \quad (173)$$

$$K_0 + p_4 = x_1 p_4 + x_2(p_1 + p_4) - x_3 p_3 \quad (174)$$

Finally, we list the integrals over Feynman parameters that arise in the box diagrams. We use the shorthand notation  $d^3 x = \prod_{i=1}^4 dx_i \delta(1 - \sum_i x_i)$ .

$$L = \int d^3 x \ln(x_1 x_3 A + x_2 x_4 B) = -\frac{11}{18} + \frac{B \ln B + A \ln A}{6(A + B)} + \frac{AB}{12(A + B)^2} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (175)$$

$$L_1 = \int d^3 x \frac{1}{x_1 x_3 A + x_2 x_4 B} = \frac{1}{2(A + B)} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (176)$$

$$L_{1A} = \int d^3 x \frac{(x_1, x_3)}{x_1 x_3 A + x_2 x_4 B} = \frac{\ln(A/B)}{2(A + B)} + \frac{B}{4(A + B)^2} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (177)$$

$$L_{1B} = \int d^3 x \frac{(x_2, x_4)}{x_1 x_3 A + x_2 x_4 B} = \frac{\ln(B/A)}{2(A + B)} + \frac{A}{4(A + B)^2} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (178)$$

$$L_A = \int d^3x \frac{x_1 x_3}{x_1 x_3 A + x_2 x_4 B} = \frac{1}{6(A+B)} + \frac{B \ln(A/B)}{3(A+B)^2} + \frac{B(B-A)}{12(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (179)$$

$$L_B = \int d^3x \frac{x_2 x_4}{x_1 x_3 A + x_2 x_4 B} = \frac{1}{6(A+B)} + \frac{A \ln(B/A)}{3(A+B)^2} + \frac{A(A-B)}{12(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (180)$$

$$\begin{aligned} L_C &= \int d^3x \frac{(x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_1)}{x_1 x_3 A + x_2 x_4 B} \\ &= -\frac{1}{6(A+B)} + \frac{(A-B) \ln(A/B)}{6(A+B)^2} + \frac{AB}{6(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (181)$$

$$L_{2B} = \int d^3x \frac{(x_2^2, x_4^2)}{x_1 x_3 A + x_2 x_4 B} = \frac{1 - \ln(A/B)}{6(A+B)} - \frac{A \ln(A/B)}{3(A+B)^2} + \frac{A^2}{6(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (182)$$

$$L_{2A} = \int d^3x \frac{(x_1^2, x_3^2)}{x_1 x_3 A + x_2 x_4 B} = \frac{1 - \ln(B/A)}{6(A+B)} - \frac{B \ln(B/A)}{3(A+B)^2} + \frac{B^2}{6(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (183)$$

$$\begin{aligned} L_{AB} &= \int d^3x \frac{x_1 x_2 x_3 x_4}{(x_1 x_3 A + x_2 x_4 B)^2} \\ &= \frac{1}{2(A+B)^2} + \frac{(B-A) \ln(A/B)}{2(A+B)^3} + \frac{A^2 + B^2 - 4AB}{12(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (184)$$

$$\begin{aligned} L_{AA} &= \int d^3x \frac{x_1^2 x_3^2}{(x_1 x_3 A + x_2 x_4 B)^2} \\ &= \frac{A - 2B}{6A(A+B)^2} + \frac{B(5A-B) \ln(A/B)}{6A(A+B)^3} + \frac{B(2B-A)}{6(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (185)$$

$$\begin{aligned} L_{CA} &= \int d^3x \frac{(x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_1) x_1 x_3}{(x_1 x_3 A + x_2 x_4 B)^2} \\ &= \frac{B - 2A}{6A(A+B)^2} - \frac{(5B-A) \ln(A/B)}{6(A+B)^3} + \frac{B(2A-B)}{6(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (186)$$

$$\begin{aligned} L_{2AB} &= \int d^3x \frac{(x_1^2, x_3^2) x_2 x_4}{(x_1 x_3 A + x_2 x_4 B)^2} \\ &= \frac{A + 4B}{6B(A+B)^2} + \frac{(A-5B) \ln(B/A)}{6(A+B)^3} + \frac{B(B-2A)}{6(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (187)$$

$$\begin{aligned} L_{2AA} &= \int d^3x \frac{(x_1^2, x_3^2) x_1 x_3}{(x_1 x_3 A + x_2 x_4 B)^2} \\ &= -\frac{B}{2A(A+B)^2} + \frac{(A^2 + 5AB + 3B^2) \ln(A/B)}{6A(A+B)^3} + \frac{B^2}{2(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (188)$$

$$\begin{aligned} L_{3A} &= \int d^3x \frac{x_2^2 x_3}{x_1 x_3 A + x_2 x_4 B} \\ &= -\frac{A}{8(A+B)^2} + \frac{(2B^2 - 5AB - A^2) \ln(A/B)}{24(A+B)^2} + \frac{A^2 B}{8(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (189)$$

## D Evaluation of Eq(93)

The numerator in the integrand, after introduction of Schwinger parameters and the appropriate shift of  $q$  can be replaced as

$$\begin{aligned} K_{61}^\wedge K_{25}^\vee K_{35}^\wedge K_{64}^\vee &\rightarrow [p_3^+ q^\wedge + x_2 K_{23}^\wedge + x_1 K_{34}^\wedge] [p_4^+ q^\vee + x_3 K_{34}^\vee + x_2 K_{41}^\vee] \\ &\quad \times [p_1^+ q^\wedge + x_3 K_{12}^\wedge + x_4 K_{41}^\wedge] [p_2^+ q^\vee + x_4 K_{23}^\vee + x_1 K_{12}^\vee] \\ &\rightarrow p_1^+ p_2^+ p_3^+ p_4^+ q^\wedge q^\vee q^\wedge q^\vee - p_1^+ p_2^+ p_3^+ p_4^+ H^2/4 \end{aligned}$$



$$\begin{aligned}
& +p_1^+p_2^+(q^\wedge q^\vee - x_1x_3p_{12}^2/2)[x_2K_{23}^\wedge + x_1K_{34}^\wedge][x_3K_{34}^\vee + x_2K_{41}^\vee] \\
& +p_1^+p_4^+(q^\wedge q^\vee - x_2x_4p_{14}^2/2)[x_2K_{23}^\wedge + x_1K_{34}^\wedge][x_4K_{23}^\vee + x_1K_{12}^\vee] \\
& +p_3^+p_4^+(q^\wedge q^\vee - x_1x_3p_{12}^2/2)[x_3K_{12}^\wedge + x_4K_{41}^\wedge][x_4K_{23}^\vee + x_1K_{12}^\vee] \\
& +p_2^+p_3^+(q^\wedge q^\vee - x_2x_4p_{14}^2/2)[x_3K_{12}^\wedge + x_4K_{41}^\wedge][x_3K_{34}^\vee + x_2K_{41}^\vee]
\end{aligned} \tag{190}$$

The second term in the first line of the last equality came from the quantity

$$-\frac{1}{4}p_1^+p_2^+p_3^+p_4^+[(x_1x_3p_{12}^2)^2 + (x_2x_4p_{14}^2)^2] + x_1x_2x_3x_4[K_{41}^\wedge K_{12}^\vee K_{23}^\wedge K_{34}^\vee + K_{34}^\wedge K_{41}^\vee K_{12}^\wedge K_{23}^\vee]$$

which can be greatly simplified using

$$K_{41}^\wedge K_{12}^\vee K_{23}^\wedge K_{34}^\vee = -\frac{p_3^+}{p_1^+}K_{41}^\wedge K_{12}^\vee K_{12}^\wedge K_{41}^\vee = -\frac{1}{4}p_1^+p_2^+p_3^+p_4^+p_{12}^2p_{14}^2 \tag{191}$$

so it becomes

$$-\frac{1}{4}p_1^+p_2^+p_3^+p_4^+(x_1x_3p_{12}^2 + x_2x_4p_{14}^2)^2 \equiv -\frac{1}{4}p_1^+p_2^+p_3^+p_4^+H^2 \tag{192}$$

Putting in the rest of the factors and doing the  $q$  integration yields the  $x$  integral

$$\begin{aligned}
& \int d^3x \frac{1}{16\pi^2} \left\{ -(\ln(H\delta e^\gamma) + 2) \right. \\
& \quad + x_2x_4p_{14}^2 \left[ \frac{(x_2K_{23}^\wedge + x_1K_{34}^\wedge)(x_3K_{34}^\vee + x_2K_{41}^\vee)}{p_3^+p_4^+H^2} + \frac{(x_3K_{12}^\wedge + x_4K_{41}^\wedge)(x_4K_{23}^\vee + x_1K_{12}^\vee)}{p_1^+p_2^+H^2} \right] \\
& \quad \left. + x_1x_3p_{12}^2 \left[ \frac{(x_2K_{23}^\wedge + x_1K_{34}^\wedge)(x_4K_{23}^\vee + x_1K_{12}^\vee)}{p_2^+p_3^+H^2} + \frac{(x_3K_{12}^\wedge + x_4K_{41}^\wedge)(x_3K_{34}^\vee + x_2K_{41}^\vee)}{p_1^+p_4^+H^2} \right] \right\} \\
& = \frac{1}{16\pi^2} \left\{ -\frac{1}{6}(\ln(\delta e^\gamma) + 2) - L(p_{12}^2, p_{14}^2) \right. \\
& \quad - B \frac{\partial}{\partial B} \int d^3x \left[ \frac{(x_2K_{23}^\wedge + x_1K_{34}^\wedge)(x_3K_{34}^\vee + x_2K_{41}^\vee)}{p_3^+p_4^+H} + \frac{(x_3K_{12}^\wedge + x_4K_{41}^\wedge)(x_4K_{23}^\vee + x_1K_{12}^\vee)}{p_1^+p_2^+H} \right] \\
& \quad \left. - A \frac{\partial}{\partial A} \int d^3x \left[ \frac{(x_2K_{23}^\wedge + x_1K_{34}^\wedge)(x_4K_{23}^\vee + x_1K_{12}^\vee)}{p_2^+p_3^+H} + \frac{(x_3K_{12}^\wedge + x_4K_{41}^\wedge)(x_3K_{34}^\vee + x_2K_{41}^\vee)}{p_1^+p_4^+H} \right] \right\} \Big|_{\substack{A=p_{12}^2 \\ B=p_{14}^2}} \tag{193} \\
& = \frac{1}{16\pi^2} \left\{ -\frac{1}{6}(\ln(\delta e^\gamma) + 2) - L(p_{12}^2, p_{14}^2) - 2p_{12}^2p_{14}^2L_{AB}(p_{12}^2, p_{14}^2) \right. \\
& \quad - B \frac{\partial}{\partial B} \left\{ L_{2B}(A, B) \left[ \frac{K_{23}^\wedge K_{41}^\vee}{p_3^+p_4^+} + \frac{K_{41}^\wedge K_{23}^\vee}{p_1^+p_2^+} \right] + L_C(A, B) \left[ \frac{K_{34}^\wedge K_{41}^\vee + K_{23}^\wedge K_{34}^\vee}{p_3^+p_4^+} + \frac{K_{12}^\wedge K_{23}^\vee + K_{41}^\wedge K_{12}^\vee}{p_1^+p_2^+} \right] \right\} \Big|_{\substack{A=p_{12}^2 \\ B=p_{14}^2}} \\
& \quad - A \frac{\partial}{\partial A} \left\{ L_{2A}(A, B) \left[ \frac{K_{34}^\wedge K_{12}^\vee}{p_2^+p_3^+} + \frac{K_{12}^\wedge K_{34}^\vee}{p_1^+p_4^+} \right] + L_C(A, B) \left[ \frac{K_{23}^\wedge K_{12}^\vee + K_{34}^\wedge K_{23}^\vee}{p_2^+p_3^+} + \frac{K_{12}^\wedge K_{41}^\vee + K_{41}^\wedge K_{34}^\vee}{p_1^+p_4^+} \right] \right\} \Big|_{\substack{A=p_{12}^2 \\ B=p_{14}^2}} \\
& = \frac{1}{16\pi^2} \left\{ -\frac{1}{6}(\ln(\delta e^\gamma) + 2) - L(p_{12}^2, p_{14}^2) - 2p_{12}^2p_{14}^2L_{AB}(p_{12}^2, p_{14}^2) \right. \\
& \quad + p_{14}^2 \left\{ L_{2AA}(p_{14}^2, p_{12}^2) \left[ \frac{K_{23}^\wedge K_{41}^\vee}{p_3^+p_4^+} + \frac{K_{41}^\wedge K_{23}^\vee}{p_1^+p_2^+} \right] + L_{CA}(p_{14}^2, p_{12}^2) \left[ \frac{K_{34}^\wedge K_{41}^\vee + K_{23}^\wedge K_{34}^\vee}{p_3^+p_4^+} + \frac{K_{12}^\wedge K_{23}^\vee + K_{41}^\wedge K_{12}^\vee}{p_1^+p_2^+} \right] \right\} \\
& \quad \left. + p_{12}^2 \left\{ L_{2AA}(p_{12}^2, p_{14}^2) \left[ \frac{K_{34}^\wedge K_{12}^\vee}{p_2^+p_3^+} + \frac{K_{12}^\wedge K_{34}^\vee}{p_1^+p_4^+} \right] + L_{CA}(p_{12}^2, p_{14}^2) \left[ \frac{K_{23}^\wedge K_{12}^\vee + K_{34}^\wedge K_{23}^\vee}{p_2^+p_3^+} + \frac{K_{12}^\wedge K_{41}^\vee + K_{41}^\wedge K_{34}^\vee}{p_1^+p_4^+} \right] \right\} \right\} \tag{194}
\end{aligned}$$

The various  $L$ 's in this formula are listed in Appendix B. Eq. 194 can be algebraically rearranged and cast into a nicer form. Putting in the coupling constant we find for the contribution of (93)

$$\begin{aligned}
& -\frac{g^2}{8\pi^2} A_{\wedge\vee\wedge\vee}^{\text{tree}} \left\{ \left[ \log^2 \frac{p_{12}^2}{p_{14}^2} + \pi^2 \right] \left[ -\frac{1}{32} \frac{p_{12}^2{}^2}{(p_{14}^2 + p_{12}^2)^2} + \frac{1}{16} \frac{p_{12}^2{}^3}{(p_{14}^2 + p_{12}^2)^3} - \frac{1}{32} \frac{p_{12}^2{}^4}{(p_{14}^2 + p_{12}^2)^4} \right] \right. \\
& + \log \frac{p_{12}^2}{p_{14}^2} \left[ \frac{1}{48} \frac{p_{12}^2}{p_{14}^2 + p_{12}^2} - \frac{3}{32} \frac{p_{12}^2{}^2}{(p_{14}^2 + p_{12}^2)^2} + \frac{1}{16} \frac{p_{12}^2{}^3}{(p_{14}^2 + p_{12}^2)^3} \right] + \frac{1}{32} \frac{p_{12}^2}{p_{14}^2 + p_{12}^2} - \frac{1}{32} \frac{p_{12}^2{}^2}{(p_{14}^2 + p_{12}^2)^2} \Big\} \\
& + \frac{g^4}{8\pi^2} \left\{ \log \frac{p_{12}^2}{p_{14}^2} \left( \left[ -\frac{1}{48} \frac{p_{12}^2(k_3^+ - k_2^+)(k_3^+ - k_0^+)}{(k_2^+ - k_0^+)p_{14}^2(k_1^+ - k_0^+)} + \frac{1}{48} \frac{p_{12}^2(k_3^+ - k_2^+)(k_3^+ - k_0^+)}{(k_2^+ - k_0^+)p_{14}^2(k_1^+ - k_2^+)} - \frac{1}{48} \frac{k_3^+ - k_0^+}{k_1^+ - k_2^+} \right. \right. \\
& + \frac{1}{48} \frac{k_3^+ - k_1^+}{k_3^+ - k_2^+} + \frac{2k_2^+ - 3k_3^+ + k_0^+}{48k_3^+ - 48k_2^+} \Big] + \frac{1}{24} \left[ \frac{1}{(k_2^+ - k_0^+)p_{14}^2(k_1^+ - k_0^+)} - \frac{k_3^+ - k_0^+}{p_{14}^2(k_3^+ - k_2^+)(k_2^+ - k_0^+)(k_1^+ - k_2^+)} \right. \\
& + \frac{k_3^+ - k_1^+}{p_{14}^2(k_3^+ - k_2^+)^2(k_3^+ - k_0^+)} - \frac{2k_3^+ - k_2^+ - k_0^+}{p_{14}^2(k_3^+ - k_2^+)^2(k_3^+ - k_0^+)} \Big] K_{32}^{\wedge} K_{43}^{\vee} \Big) - \frac{1}{24} \log p_{14}^2 \delta e^\gamma + \frac{1}{144} \Big\}
\end{aligned}$$

## E Complete Results of the Two Model Boxes

There are basically two model integrands that has an adjacent pair of vertices with the same helicity.

$$\mathcal{N} = K_{61}^{\wedge} K_{25}^{\vee} K_{5,3}^{\vee} K_{4,6}^{\wedge} \quad \text{or} \quad \mathcal{N} = K_{61}^{\wedge} K_{25}^{\wedge} K_{5,3}^{\vee} K_{4,6}^{\vee}$$

Assume for the moment that  $\mathcal{R} = 1$ . These diagram can be reduced into triangles using Eq.(92). The first contribution is

$$\frac{1}{8\pi^2} g^4 K_{61}^{\wedge} K_{25}^{\vee} K_{5,3}^{\vee} K_{4,6}^{\wedge} \times$$

$$\begin{aligned}
& k_0^+ < q^+ < k_3^+ \\
& - \frac{1}{8} \frac{(q^+ - k_2^+)^2(k_0^+ - k_1^+)(k_3^+ - k_0^+)}{k_0^+ - k_2^+} \log(H_s \delta e^\gamma) + \frac{1}{8} \frac{(k_3^+ - q^+)^2(k_1^+ - k_2^+)(k_0^+ - k_1^+)}{k_3^+ - k_1^+} \log(H_d \delta e^\gamma) \\
& + \frac{1}{4} \frac{(q^+ - k_1^+)(k_3^+ - q^+)}{(k_3^+ - k_1^+)p_{14}^2} K_{14}^{\wedge} K_{32}^{\vee} \log(H_d \delta e^\gamma) + \frac{1}{8} \frac{(q^+ - k_1^+)^2(k_3^+ - k_2^+)(k_3^+ - k_0^+)}{k_3^+ - k_1^+} \log(H_u \delta e^\gamma) \\
& - \frac{1}{4} \frac{(q^+ - k_1^+)(k_3^+ - q^+)}{(k_3^+ - k_1^+)p_{14}^2} K_{14}^{\wedge} K_{32}^{\vee} \log(H_u \delta e^\gamma)
\end{aligned}$$

$$\begin{aligned}
& k_3^+ < q^+ < k_1^+ \\
& + \left[ \frac{1}{8} \frac{(k_1^+ - k_2^+)(k_0^+ - k_1^+)^2(k_3^+ - k_2^+)}{k_0^+ - k_2^+} - \frac{1}{4} \frac{(k_1^+ - k_2^+)(k_3^+ - k_2^+)(k_0^+ - k_1^+)(q^+ - k_1^+)}{k_0^+ - k_2^+} \right. \\
& - \frac{1}{8} \frac{(k_0^+ - k_1^+)(k_3^+ - k_2^+)(k_3^+ - k_0^+ - k_1^+ + k_2^+)(q^+ - k_1^+)^2}{(k_0^+ - k_2^+)(k_3^+ - k_1^+)} \Big] \log(H_s \delta e^\gamma) \\
& + \frac{1}{4} \frac{(q^+ - k_1^+)(k_3^+ - q^+)}{(k_3^+ - k_1^+)p_{14}^2} K_{14}^{\wedge} K_{32}^{\vee} \log(H_s \delta e^\gamma) + \frac{1}{4} \frac{(q^+ - k_1^+)(k_3^+ - q^+)}{(k_3^+ - k_1^+)p_{14}^2} K_{14}^{\wedge} K_{32}^{\vee} \log(H_t \delta e^\gamma) \\
& - \frac{1}{8} \frac{(k_3^+ - q^+)^2(k_1^+ - k_2^+)(k_0^+ - k_1^+)}{k_3^+ - k_1^+} \log(H_d \delta e^\gamma) - \frac{1}{4} \frac{(q^+ - k_1^+)(k_3^+ - q^+)}{(k_3^+ - k_1^+)p_{14}^2} K_{14}^{\wedge} K_{32}^{\vee} \log(H_d \delta e^\gamma) \\
& + \frac{1}{8} \frac{(q^+ - k_1^+)^2(k_3^+ - k_2^+)(k_3^+ - k_0^+)}{k_3^+ - k_1^+} \log(H_u \delta e^\gamma) - \frac{1}{4} \frac{(q^+ - k_1^+)(k_3^+ - q^+)}{(k_3^+ - k_1^+)p_{14}^2} K_{14}^{\wedge} K_{32}^{\vee} \log(H_u \delta e^\gamma)
\end{aligned}$$

$$\begin{aligned}
& k_1^+ < q^+ < k_2^+ \\
& + \frac{1}{8} \frac{(k_1^+ - k_2^+)(k_0^+ - q^+)^2(k_3^+ - k_2^+)}{k_0^+ - k_2^+} \log(H_s \delta e^\gamma) - \frac{1}{8} \frac{(k_3^+ - q^+)^2(k_1^+ - k_2^+)(k_0^+ - k_1^+)}{k_3^+ - k_1^+} \log(H_d \delta e^\gamma) \\
& - \frac{1}{4} \frac{(k_3^+ - q^+)(q^+ - k_1^+)}{(k_3^+ - k_1^+)p_{14}^2} K_{14}^\wedge K_{32}^\vee \log(H_d \delta e^\gamma) - \frac{1}{8} \frac{(q^+ - k_1^+)^2(k_3^+ - k_2^+)(k_3^+ - k_0^+)}{k_3^+ - k_1^+} \log(H_u \delta e^\gamma) \\
& + \frac{1}{4} \frac{(k_3^+ - q^+)(q^+ - k_1^+)}{(k_3^+ - k_1^+)p_{14}^2} K_{14}^\wedge K_{32}^\vee \log(H_u \delta e^\gamma)
\end{aligned}$$

Assuming its coefficient to be  $-C - A \frac{1}{(q^+ - k_1^+)^2} + B \frac{1}{(q^+ - k_1^+)}$  or  $-C - A \frac{1}{(q^+ - k_3^+)^2} + B \frac{1}{(q^+ - k_3^+)}$ , its contribution will be

$$\begin{aligned}
& \frac{1}{8\pi^2} g^4 \left\{ \left[ \frac{C}{36} - \frac{C}{24} \log(p_{12}^2 \delta e^\gamma) \right] (-k_3^+ + k_0^+)(-k_0^+ + k_1^+)(k_2^+ - k_3^+)(k_2^+ - k_1^+) \right. \\
& + \left[ \frac{C}{24} \frac{2k_3^+ k_1^+ - k_0^+ k_1^+ - k_2^+ k_1^+ - k_3^+ k_0^+ - k_3^+ k_2^+ + k_2^{+2} + k_0^{+2}}{p_{14}^2} - \frac{C}{24} \frac{(k_1^+ - k_3^+)^2}{p_{14}^2} \log \frac{p_{12}^2}{p_{14}^2} \right. \\
& \left. \left. + \frac{B}{8} \frac{k_1^+ + k_3^+ - k_0^+ - k_2^+}{p_{14}^2} + \frac{A}{4p_{14}^2} \log \frac{p_{12}^2}{p_{14}^2} \mp \frac{B}{8} \frac{k_1^+ - k_3^+}{p_{14}^2} \log \frac{p_{12}^2}{p_{14}^2} \right] K_{14}^\wedge K_{32}^\vee + \text{infra red terms} \right\}
\end{aligned}$$

And it's impossible for this diagram to have poles at  $q^+ = k_0^+$  or  $k_2^+$ . The reader can work out the infrared terms by partial fractioning the coefficient of each logarithm, and taking only the pole terms (but be very careful when applying these results to actual boxes, you might need to take a conjugation occasionally).

The second case is very similar:

$$\frac{1}{8\pi^2} g^4 K_{61}^\wedge K_{25}^\wedge K_{5,3}^\vee K_{4,6}^\vee \times$$

$$\begin{aligned}
& k_0^+ < q^+ < k_3^+ \\
& + \frac{1}{4} \frac{(q^+ - k_2^+)(k_0^+ - q^+)}{(k_0^+ - k_2^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \log(H_s \delta e^\gamma) + \frac{1}{8} \frac{(k_0^+ - q^+)^2(k_1^+ - k_2^+)(k_3^+ - k_2^+)}{k_0^+ - k_2^+} \log(H_r \delta e^\gamma) \\
& - \frac{1}{4} \frac{(q^+ - k_2^+)(k_0^+ - q^+)}{(k_0^+ - k_2^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \log(H_r \delta e^\gamma) \\
& k_3^+ < q^+ < k_1^+ \\
& + \frac{1}{4} \frac{(q^+ - k_2^+)(-q^+ + k_0^+)}{(k_0^+ - k_2^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \log(H_s \delta e^\gamma) \\
& + \left[ \frac{1}{8} \frac{(k_0^+ - k_1^+)^2(k_3^+ - k_0^+)(k_3^+ - k_2^+)}{k_3^+ - k_1^+} - \frac{1}{4} \frac{(k_3^+ - k_0^+)(k_3^+ - k_2^+)(k_0^+ - k_1^+)(-q^+ + k_0^+)}{k_3^+ - k_1^+} \right. \\
& \left. + \frac{1}{8} \frac{(k_0^+ - k_1^+)(k_3^+ - k_2^+)(k_3^+ - k_1^+ - k_0^+ + k_2^+)(-q^+ + k_0^+)^2}{(k_0^+ - k_2^+)(k_3^+ - k_1^+)} \right] \log(H_t \delta e^\gamma) \\
& + \frac{1}{4} \frac{(q^+ - k_2^+)(-q^+ + k_0^+)}{(k_0^+ - k_2^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \log(H_t \delta e^\gamma) - \frac{1}{8} \frac{(q^+ - k_2^+)^2(k_0^+ - k_1^+)(k_3^+ - k_0^+)}{k_0^+ - k_2^+} \log(H_l \delta e^\gamma) \\
& - \frac{1}{4} \frac{(q^+ - k_2^+)(-q^+ + k_0^+)}{(k_0^+ - k_2^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \log(H_l \delta e^\gamma) + \frac{1}{8} \frac{(-q^+ + k_0^+)^2(k_1^+ - k_2^+)(k_3^+ - k_2^+)}{k_0^+ - k_2^+} \log(H_r \delta e^\gamma) \\
& - \frac{1}{4} \frac{(q^+ - k_2^+)(-q^+ + k_0^+)}{(k_0^+ - k_2^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \log(H_r \delta e^\gamma)
\end{aligned}$$

$$\begin{aligned}
& k_1^+ < q^+ < k_2^+ \\
& + \frac{1}{4} \frac{(q^+ - k_2^+)(-q^+ + k_0^+)}{(k_0^+ - k_2^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \log(H_s \delta e^\gamma) - \frac{1}{8} \frac{(q^+ - k_2^+)^2 (k_0^+ - k_1^+)(k_3^+ - k_0^+)}{k_0^+ - k_2^+} \log(H_l \delta e^\gamma) \\
& - \frac{1}{4} \frac{(q^+ - k_2^+)(-q^+ + k_0^+)}{(k_0^+ - k_2^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \log(H_l \delta e^\gamma)
\end{aligned}$$

Assuming its coefficient to be  $-C - A \frac{1}{(q^+ - k_0^+)^2} + B \frac{1}{(q^+ - k_0^+)}$  or  $-C - A \frac{1}{(q^+ - k_2^+)^2} + B \frac{1}{(q^+ - k_2^+)}$ , its contribution will be:

$$\begin{aligned}
& \frac{1}{8\pi^2} g^4 \left\{ \left[ \frac{C}{36} - \frac{C}{24} \log(p_{14}^2 \delta e^\gamma) \right] (k_3^+ - k_2^+)(k_1^+ - k_2^+)(k_0^+ - k_1^+)(k_3^+ - k_0^+) \right. \\
& + \left[ -\frac{C}{24} \frac{(-k_1^{+2} - k_3^{+2} + k_1^+ k_0^+ + k_0^+ k_3^+ + k_1^+ k_2^+ + k_2^+ k_3^+ - 2k_0^+ k_2^+)}{p_{12}^2} + \frac{1}{24} \frac{(k_0^+ - k_2^+)^2}{p_{12}^2} \log \frac{p_{12}^2}{p_{14}^2} \right. \\
& \left. \left. - \frac{B - k_0^+ + k_1^+ + k_3^+ - k_2^+}{p_{12}^2} - \frac{1}{4p_{12}^2} \log \frac{p_{12}^2}{p_{14}^2} \pm \frac{1}{8} \cdot \frac{k_0^+ - k_2^+}{p_{12}^2} \log \frac{p_{12}^2}{p_{14}^2} \right] K_{21}^\wedge K_{43}^\vee + \text{infra red terms} \right\}
\end{aligned}$$

And it's impossible for this diagram to have poles at  $q^+ = k_1^+$  or  $k_3^+$ .

## F Complete Results of the Second Term of Eq.(104)

$$\begin{aligned}
& \frac{1}{8\pi^2} g^4 A \log \frac{p_{12}^2}{p_{14}^2} \times \left\{ \right. \\
& - \frac{1}{8} \frac{(k_3^+ - k_2^+)(k_3^+ - k_0^+)((k_2^+ - k_1^+)p_{14}^2 + p_{12}^2 k_3^+ - p_{12}^2 k_0^+)}{(k_3^+ - k_1^+)(p_{14}^2 + p_{12}^2)} + \frac{1}{4} \frac{(k_3^+ - k_2^+)p_{14}^2 + p_{12}^2 k_0^+ - k_1^+ p_{12}^2}{(k_3^+ - k_1^+)p_{14}^2 (p_{14}^2 + p_{12}^2)} K_{14}^\wedge K_{32}^\vee \left. \right\} \\
& \frac{1}{8\pi^2} g^4 A \left[ \log^2 \frac{p_{12}^2}{p_{14}^2} + \pi^2 \right] \times \left\{ -\frac{1}{16} \frac{p_{14}^2 p_{12}^2 (-k_0^+ + k_3^+)(k_3^+ - k_2^+)(k_3^+ - k_0^+ + k_1^+ - k_2^+)}{(p_{14}^2 + p_{12}^2)^2 (k_3^+ - k_1^+)} \right. \\
& \left. - \frac{1}{8} \frac{p_{12}^2 ((-k_0^+ + 2k_3^+ - k_2^+)p_{14}^2 + p_{12}^2 (k_3^+ - k_1^+))}{(k_3^+ - k_1^+)(p_{14}^2 + p_{12}^2)^2 p_{14}^2} K_{14}^\wedge K_{32}^\vee \right\} \\
& \frac{1}{8\pi^2} g^4 B \log \frac{p_{12}^2}{p_{14}^2} \times \left\{ \right. \\
& - \frac{1}{16} \frac{p_{12}^2 (k_2^+ - k_1^+)(3k_2^{+2} - 6k_3^+ k_2^+ - 2k_1^+ k_2^+ + 2k_0^+ k_2^+ + 3k_3^{+2} - 4k_0^+ k_3^+ + 4k_3^+ k_1^+ - k_1^{+2} + k_0^{+2})}{p_{14}^2 + p_{12}^2} \\
& + \frac{1}{8} \frac{k_3^{+2} + 2k_3^+ k_1^+ - 2k_0^+ k_3^+ - 2k_3^+ k_2^+ - k_1^{+2} + k_2^{+2} + k_0^{+2}}{(k_3^+ - k_2^+)(p_{14}^2 + p_{12}^2)} K_{32}^\wedge K_{43}^\vee \\
& + \frac{1}{8} \frac{p_{12}^2 (k_2^+ - k_1^+)(k_3^+ - k_0^+ + k_1^+ - k_2^+)^2}{(p_{14}^2 + p_{12}^2)^2} - \frac{1}{4} \frac{p_{12}^2 (k_3^+ - k_0^+ + k_1^+ - k_2^+)^2}{(k_3^+ - k_2^+)(p_{14}^2 + p_{12}^2)^2} K_{32}^\wedge K_{43}^\vee \\
& + \frac{1}{16} (k_3^+ - k_1^+)^2 (k_3^+ - k_2^+) - \frac{1}{16} (k_3^+ - k_1^+)(k_3^+ - k_0^+)(k_3^+ - k_2^+) \\
& + \frac{1}{8} \frac{k_1^+ - k_3^+}{p_{14}^2} K_{14}^\wedge K_{32}^\vee + \frac{1}{16} (k_3^+ - k_2^+)^2 (-k_0^+ + k_2^+) \left. \right\} \\
& \frac{1}{8\pi^2} g^4 B \left[ \log^2 \frac{p_{12}^2}{p_{14}^2} + \pi^2 \right] \times \left\{ -\frac{1}{16} \frac{p_{12}^2 (k_3^+ - k_2^+)^2 (k_2^+ - k_1^+)}{p_{14}^2 + p_{12}^2} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{16} \frac{p_{12}^2 (k_2^+ - k_1^+) (2k_2^{+2} + 2k_0^+ k_2^+ - 4k_3^+ k_2^+ - 2k_1^+ k_2^+ - 3k_0^+ k_3^+ + 3k_3^+ k_1^+ + 2k_3^{+2} + k_0^{+2} - k_0^+ k_1^+)}{(p_{14}^2 + p_{12}^2)^2} \\
& - \frac{1}{8} \frac{(-k_1^+ k_2^+ + k_0^+ k_2^+ - k_0^+ k_1^+ + k_3^{+2} - 2k_0^+ k_3^+ + 2k_3^+ k_1^+ - 2k_3^+ k_2^+ + k_0^{+2} + k_2^{+2}) p_{12}^2}{(k_3^+ - k_2^+) (p_{14}^2 + p_{12}^2)^2} K_{32}^\wedge K_{43}^\vee \\
& - \frac{1}{16} \frac{p_{12}^3 (k_2^+ - k_1^+) (k_3^+ - k_0^+ + k_1^+ - k_2^+)^2}{(p_{14}^2 + p_{12}^2)^3} + \frac{1}{8} \frac{p_{12}^2 (k_3^+ - k_0^+ + k_1^+ - k_2^+)^2}{(k_3^+ - k_2^+) (p_{14}^2 + p_{12}^2)^3} K_{32}^\wedge K_{43}^\vee \Big\} \\
& \frac{1}{8\pi^2} g^4 B \times \left\{ - \frac{1}{16} \frac{p_{12}^2 (k_2^+ - k_1^+) (k_3^+ - k_0^+ + k_1^+ - k_2^+)^2}{p_{14}^2 + p_{12}^2} + \frac{1}{8} \frac{(k_3^+ - k_0^+ + k_1^+ - k_2^+)^2}{(k_3^+ - k_2^+) (p_{14}^2 + p_{12}^2)} K_{32}^\wedge K_{43}^\vee \right. \\
& - \frac{1}{16} (k_3^+ - k_1^+)^2 (k_3^+ - k_2^+) + \frac{1}{16} (k_3^+ - k_1^+) (k_3^+ - k_2^+) (3k_3^+ - k_0^+ - 2k_2^+) \\
& \left. + \frac{1}{8} \frac{k_3^+ - k_1^+}{p_{14}^2} K_{14}^\wedge K_{32}^\vee - \frac{1}{16} (k_3^+ - k_2^+)^2 (2k_3^+ - k_0^+ - k_2^+) - \frac{1}{8} \frac{2k_3^+ - k_0^+ - k_2^+}{p_{14}^2} K_{14}^\wedge K_{32}^\vee \right\}
\end{aligned}$$

The results for other pole locations can be obtained from this by rotational symmetry and conjugation.

If we put these terms from each diagram together, there is usually some nice simplifications.

For the fifth box of fig.7:

$$- \frac{g^2}{8\pi^2} A_{\wedge\vee\wedge\vee}^{tree} \left\{ \left[ \log^2 \frac{p_{12}^2}{p_{14}^2} + \pi^2 \right] \left[ \frac{1}{2} \frac{p_{12}^2}{(p_{14}^2 + p_{12}^2)^2} - \frac{p_{12}^3}{(p_{14}^2 + p_{12}^2)^3} \right] + \log \frac{p_{12}^2}{p_{14}^2} \left[ 2 \frac{p_{12}^2}{(p_{14}^2 + p_{12}^2)^2} \right] - \frac{p_{12}^2}{p_{14}^2 + p_{12}^2} \right\}$$

And also a piece that doesn't fall into a tree:

$$\begin{aligned}
& \frac{1}{8\pi^2} g^4 \log \frac{p_{12}^2}{p_{14}^2} \times \left\{ \left[ 2 \frac{p_{12}^2 (-k_0^+ + k_2^+)}{p_{14}^2 (k_0^+ - k_1^+)} + 2 \frac{p_{12}^2 (k_3^+ - k_2^+)}{p_{14}^2 (k_3^+ - k_1^+)} + 2 \frac{p_{12}^2 (k_3^+ - k_2^+)}{p_{14}^2 (k_2^+ - k_1^+)} - 2 \frac{(k_3^+ - k_0^+) (-k_0^+ + k_2^+)}{(k_3^+ - k_1^+)^2} \right. \right. \\
& + 2 \frac{(-k_0^+ + k_2^+) (2k_3^+ - k_0^+ - k_2^+)}{(k_3^+ - k_2^+) (k_3^+ - k_1^+)} - 2 \frac{-k_0^+ + k_2^+}{k_2^+ - k_1^+} \Big] \\
& + \left[ -4 \frac{-k_0^+ + k_2^+}{p_{14}^2 (k_3^+ - k_2^+) (k_3^+ - k_0^+) (k_0^+ - k_1^+)} - 4 \frac{2k_3^+ - k_0^+ - k_2^+}{p_{14}^2 (k_3^+ - k_2^+) (k_3^+ - k_1^+)^2} + 4 \frac{1}{p_{14}^2 (k_3^+ - k_2^+) (k_3^+ - k_0^+)} \right. \\
& + 4 \frac{-k_2^{+2} + k_0^+ k_2^+ + k_3^+ k_2^+ - 3k_0^+ k_3^+ + k_0^{+2} + k_3^{+2}}{(k_3^+ - k_0^+) (k_3^+ - k_2^+)^2 p_{14}^2 (k_3^+ - k_1^+)} - 4 \frac{1}{p_{14}^2 (k_3^+ - k_2^+) (k_2^+ - k_1^+)} \Big] K_{32}^\wedge K_{43}^\vee \Big\} \\
& \frac{1}{8\pi^2} g^4 \times \left\{ \left[ 2 \frac{(2k_3^+ - k_0^+ - k_2^+) p_{12}^2}{(k_3^+ - k_1^+) p_{14}^2} - 2 \frac{(-k_0^+ + k_2^+) (2k_3^+ - k_0^+ - k_2^+)}{(k_3^+ - k_1^+)^2} + 2 \frac{-k_0^+ + k_2^+}{k_3^+ - k_1^+} + 2 \right] \right. \\
& \left. + \left[ -8 \frac{2k_3^+ - k_0^+ - k_2^+}{p_{14}^2 (k_3^+ - k_1^+)^2 (k_3^+ - k_2^+)} + 8 \frac{1}{p_{14}^2 (k_3^+ - k_1^+) (k_3^+ - k_2^+)} \right] K_{32}^\wedge K_{43}^\vee \right\}
\end{aligned}$$

For the fourth box of fig.7:

$$\begin{aligned}
& - \frac{1}{8\pi^2} g^2 A_{\wedge\vee\wedge\vee}^{tree} \left\{ \left[ \log^2 \frac{p_{12}^2}{p_{14}^2} + \pi^2 \right] \left[ \frac{p_{12}^2}{p_{14}^2 + p_{12}^2} - \frac{5}{2} \frac{p_{12}^2}{(p_{14}^2 + p_{12}^2)^2} + \frac{p_{12}^3}{(p_{14}^2 + p_{12}^2)^3} - \frac{1}{2} \right] \right. \\
& \left. + \log \frac{p_{12}^2}{p_{14}^2} \left[ 4 \frac{p_{12}^2}{p_{14}^2 + p_{12}^2} - 2 \frac{p_{12}^2}{(p_{14}^2 + p_{12}^2)^2} \right] + \frac{p_{12}^2}{p_{14}^2 + p_{12}^2} \right\}
\end{aligned}$$

And also a piece that doesn't fall into a tree:

$$\frac{1}{8\pi^2} g^4 \log \frac{p_{12}^2}{p_{14}^2} \times \left\{ \right.$$

$$\begin{aligned}
& + \left[ 4 \frac{p_{12}^2(k_3^+ - k_0^+)(k_3^+ - k_2^+)}{(k_2^+ - k_0^+)p_{14}^2(k_1^+ - k_0^+)} - 4 \frac{p_{12}^2(k_3^+ - k_0^+)(k_3^+ - k_2^+)}{p_{14}^2(k_2^+ - k_0^+)(k_1^+ - k_2^+)} + 4 \frac{k_3^+ - k_0^+}{k_1^+ - k_2^+} - 4 \frac{k_3^+ - k_1^+}{k_3^+ - k_2^+} + 4 \frac{2k_3^+ - k_2^+ - k_0^+}{k_3^+ - k_2^+} \right] \\
& + \left[ -4 \frac{k_3^+ - k_2^+}{(k_2^+ - k_0^+)(k_3^+ - k_0^+)(k_1^+ - k_0^+)p_{12}^2} + 4 \frac{k_3^+ - k_2^+}{(k_2^+ - k_0^+)^2(k_3^+ - k_0^+)p_{12}^2} \right] K_{21}^\wedge K_{14}^\vee \\
& + \left[ 4 \frac{-2k_2^+ + k_0^+ + k_3^+}{(k_2^+ - k_0^+)(k_3^+ - k_0^+)(k_1^+ - k_0^+)p_{12}^2} + 4 \frac{1}{(k_2^+ - k_0^+)(k_3^+ - k_0^+)p_{12}^2} \right] K_{21}^\wedge K_{43}^\vee \\
& + \left[ -4 \frac{-2k_0^+ + k_2^+ + k_3^+}{(k_2^+ - k_0^+)(k_3^+ - k_2^+)(k_1^+ - k_2^+)p_{12}^2} - 4 \frac{1}{(k_2^+ - k_0^+)(k_3^+ - k_2^+)p_{12}^2} \right] K_{43}^\wedge K_{21}^\vee \\
& + \left[ -8 \frac{1}{p_{14}^2(k_2^+ - k_0^+)(k_1^+ - k_0^+)} + 8 \frac{k_3^+ - k_0^+}{p_{14}^2(k_3^+ - k_2^+)(k_2^+ - k_0^+)(k_1^+ - k_2^+)} - 8 \frac{k_3^+ - k_1^+}{(k_3^+ - k_0^+)(k_3^+ - k_2^+)^2 p_{14}^2} \right. \\
& \left. + 8 \frac{2k_3^+ - k_2^+ - k_0^+}{(k_3^+ - k_0^+)(k_3^+ - k_2^+)^2 p_{14}^2} \right] K_{32}^\wedge K_{43}^\vee \\
& + \left[ 4 \frac{k_3^+ - k_0^+}{(k_2^+ - k_0^+)(k_3^+ - k_2^+)(k_1^+ - k_2^+)p_{12}^2} + 4 \frac{k_3^+ - k_0^+}{(k_2^+ - k_0^+)^2(k_3^+ - k_2^+)p_{12}^2} \right] K_{43}^\wedge K_{32}^\vee \Big\} \\
& \frac{1}{8\pi^2} g^4 \times \left\{ \left[ 2 \frac{k_3^+ - k_0^+}{k_1^+ - k_2^+} - 2 \frac{k_3^+ - k_1^+}{k_3^+ - k_2^+} + 2 \frac{2k_3^+ - k_2^+ - k_0^+}{k_3^+ - k_2^+} - 2 \frac{p_{12}^2(k_3^+ - k_0^+)(k_3^+ - k_2^+)}{p_{14}^2(k_1^+ - k_2^+)(k_1^+ - k_0^+)} \right] \right. \\
& + \left[ 4 \frac{k_3^+ - k_2^+}{(k_2^+ - k_0^+)(k_3^+ - k_0^+)(k_1^+ - k_0^+)p_{12}^2} + 4 \frac{k_3^+ - k_1^+}{(k_2^+ - k_0^+)^3 p_{12}^2} - 4 \frac{(k_3^+ - k_2^+)(k_2^+ - 3k_0^+ + 2k_3^+)}{(k_2^+ - k_0^+)^3(k_3^+ - k_0^+)p_{12}^2} \right] K_{21}^\wedge K_{43}^\vee \\
& + \left[ -4 \frac{k_3^+ - k_0^+}{(k_2^+ - k_0^+)(k_3^+ - k_2^+)(k_1^+ - k_2^+)p_{12}^2} - 4 \frac{k_3^+ - k_1^+}{(k_2^+ - k_0^+)^3 p_{12}^2} + 4 \frac{(k_3^+ - k_0^+)(2k_3^+ + k_0^+ - 3k_2^+)}{(k_2^+ - k_0^+)^3(k_3^+ - k_2^+)p_{12}^2} \right] K_{43}^\wedge K_{21}^\vee \\
& \left. + 4 \frac{(-k_2^+ + k_3^+ - k_0^+ + k_1^+)(k_1^{+2} - k_2^+ k_1^+ - k_0^+ k_1^+ - k_3^+ k_2^+ + 2k_2^+ k_0^+ - k_3^+ k_0^+ + k_3^{+2})}{(k_3^+ - k_0^+)(k_1^+ - k_2^+)(k_3^+ - k_2^+)^2(k_1^+ - k_0^+)p_{14}^2} K_{32}^\wedge K_{43}^\vee \right\}
\end{aligned}$$

For the seventh box of fig.7:

$$-\frac{1}{2} \cdot \left[ -\frac{1}{8\pi^2} g^2 A_{\wedge\vee\wedge\vee}^{tree} \right]$$

And:

$$\begin{aligned}
& \frac{1}{8\pi^2} g^4 \log \frac{p_{12}^2}{p_{14}^2} \times \left\{ \right. \\
& + 4 \frac{(k_3^+ - k_0^+)(k_1^+ - k_0^+)}{(k_2^+ - k_0^+)^2(k_3^+ - k_2^+)(k_1^+ - k_2^+)p_{12}^2} K_{14}^\wedge K_{21}^\vee + 4 \frac{(k_3^+ - k_0^+)(k_3^+ - k_2^+)(-k_2^+ - k_0^+ + 2k_1^+)}{(k_1^+ - k_2^+)^2(k_3^+ - k_1^+)(k_1^+ - k_0^+)^2 p_{14}^2} K_{14}^\wedge K_{32}^\vee \\
& - 4 \frac{(2k_3^+ - k_2^+ - k_0^+)(k_1^+ - k_0^+)(k_1^+ - k_2^+)}{(k_3^+ - k_0^+)^2(k_3^+ - k_2^+)^2 p_{14}^2(k_3^+ - k_1^+)} K_{32}^\wedge K_{14}^\vee - 4 \frac{(k_1^+ - k_0^+)(k_1^+ - k_2^+)}{(k_3^+ - k_0^+)(k_3^+ - k_2^+)(k_3^+ - k_1^+)^2 p_{14}^2} K_{32}^\wedge K_{21}^\vee \\
& - 4 \frac{(k_3^+ - k_2^+)(k_3^+ - k_0^+)}{(k_1^+ - k_0^+)(k_3^+ - k_1^+)^2(k_1^+ - k_2^+)p_{14}^2} K_{14}^\wedge K_{43}^\vee + 4 \frac{(k_3^+ - k_2^+)(k_1^+ + k_3^+ - 2k_0^+)(k_1^+ - k_2^+)}{(k_3^+ - k_0^+)^2(k_1^+ - k_0^+)^2 p_{12}^2(k_2^+ - k_0^+)} K_{21}^\wedge K_{43}^\vee \\
& - 4 \frac{(k_3^+ - k_0^+)(k_1^+ - k_0^+)(-2k_2^+ + k_1^+ + k_3^+)}{(k_3^+ - k_2^+)^2(k_1^+ - k_2^+)^2 p_{12}^2(k_2^+ - k_0^+)} K_{43}^\wedge K_{21}^\vee + 4 \frac{(k_3^+ - k_2^+)(k_1^+ - k_2^+)}{(k_2^+ - k_0^+)^2(k_3^+ - k_0^+)(k_1^+ - k_0^+)p_{12}^2} K_{32}^\wedge K_{43}^\vee \Big\} \\
& + \frac{1}{8\pi^2} g^4 \times \left\{ \right. \\
& + 4 \frac{(-k_2^+ + k_3^+ - k_0^+ + k_1^+)(k_3^+ - k_2^+)(k_3^+ - k_0^+)}{p_{14}^2(k_1^+ - k_2^+)^2(k_3^+ - k_1^+)^3(k_1^+ - k_0^+)^2} .
\end{aligned}$$

$$\begin{aligned}
& (-3k_1^{+2} + 2k_3^+k_1^+ + 2k_1^+k_2^+ + 2k_0^+k_1^+ - k_3^+k_0^+ - k_3^+k_2^+ - k_2^+k_0^+)K_{14}^\wedge K_{32}^\vee \\
& + 4 \frac{(-k_2^+ + k_3^+ - k_0^+ + k_1^+)(k_1^+ - k_2^+)(k_1^+ - k_0^+)}{p_{14}^2(k_3^+ - k_1^+)^3(k_3^+ - k_2^+)^2(k_3^+ - k_0^+)^2} \cdot \\
& (k_2^+k_0^+ - 2k_3^+k_0^+ + k_0^+k_1^+ + 3k_3^{+2} - 2k_3^+k_1^+ - 2k_3^+k_2^+ + k_1^+k_2^+)K_{32}^\wedge K_{14}^\vee \\
& + 4 \frac{(k_3^+ - k_2^+)(k_1^+ - k_2^+)(-k_2^+ + k_3^+ - k_0^+ + k_1^+)}{(k_3^+ - k_0^+)^2(k_2^+ - k_0^+)^3(k_1^+ - k_0^+)^2p_{12}^2} \cdot \\
& (-2k_2^+k_0^+ + k_1^+k_2^+ + k_3^+k_2^+ + 3k_0^{+2} - 2k_0^+k_1^+ - 2k_3^+k_0^+ + k_3^+k_1^+)K_{21}^\wedge K_{43}^\vee \\
& - 4 \frac{(k_3^+ - k_0^+)(k_1^+ - k_0^+)(-k_2^+ + k_3^+ - k_0^+ + k_1^+)}{(k_3^+ - k_2^+)^2(k_2^+ - k_0^+)^3(k_1^+ - k_2^+)^2p_{12}^2} \cdot \\
& (3k_2^{+2} - 2k_1^+k_2^+ - 2k_3^+k_2^+ - 2k_2^+k_0^+ + k_0^+k_1^+ + k_3^+k_1^+ + k_3^+k_0^+)K_{43}^\wedge K_{21}^\vee \Big\}
\end{aligned}$$

## G Details on Triangle-like Diagrams with Collinear Divergences

After ascertaining that the combination of

$$\begin{aligned}
& -\frac{A}{2} \frac{K_{3,5}^\vee K_{4,3}^\wedge p_{12}^2}{q_1^2 q_2^2 q_3^2 p_{12}^2} - \frac{A}{2} \frac{K_{3,5}^\vee K_{1,4}^\wedge p_{12}^2}{q_1^2 q_2^2 q_0^2 p_{14}^2} \\
& - \frac{A}{2} \frac{K_{3,5}^\vee K_{6,4}^\wedge}{(q^+ - k_1^+)^2} \left[ \frac{(-k_1^+ + k_0^+)(-k_1^+ + q^+)}{q_1^2 q_3^2 q_0^2} + \frac{(-k_2^+ + k_1^+)(k_1^+ - q^+)}{q_1^2 q_2^2 q_3^2} \right]
\end{aligned}$$

doesn't have collinear divergence, we simply list its contribution here. Note in the results below, we have put back in the term

$$-\frac{A}{2} \frac{K_{3,5}^\vee K_{6,4}^\wedge}{(q^+ - k_1^+)^2} \frac{(k_1^+ + k_0^+)(q^+ - k_2^+) + (k_2^+ - k_1^+)(q^+ - k_2^+)}{q_0^2 q_3^2 q_2^2} - A \frac{K_{3,5}^\vee K_{6,4}^\wedge}{(q^+ - k_1^+)^2} \frac{K_{16}^\wedge K_{5,2}^\vee}{q_0^2 q_1^2 q_3^2 q_2^2}$$

(more specifically, the above contains all the triangle-like terms in Eq.(104) and the two subtractions from the first term of Eq.(104)).

$$\begin{aligned}
& \frac{1}{8\pi^2} g^4 \times \\
& k_0^+ < q^+ < k_3^+ \\
& - \frac{1}{8} \frac{(k_3^+ - k_0^+)(k_0^+ - k_1^+)(q^+ - k_2^+)^2}{(q^+ - k_1^+)^2(k_0^+ - k_2^+)} \log(H_s) + \frac{1}{4} \frac{(q^+ - k_2^+)(k_3^+ - k_1^+)}{(q^+ - k_1^+)(k_1^+ - k_2^+)p_{14}^2(k_3^+ - k_2^+)} K_{43}^\wedge K_{32}^\vee \log(H_s) \\
& + \frac{1}{8} \frac{(k_3^+ - q^+)(q^+ - k_2^+)(k_0^+ - k_1^+)}{(q^+ - k_1^+)^2} \log(H_d) - \frac{1}{4} \frac{k_3^+ - q^+}{(q^+ - k_1^+)(k_3^+ - k_2^+)p_{14}^2} K_{43}^\wedge K_{32}^\vee \log(H_d) \\
& - \frac{1}{4} \frac{q^+ - k_2^+}{p_{14}^2(k_1^+ - k_2^+)(q^+ - k_1^+)} K_{14}^\wedge K_{32}^\vee \log(H_u) - \frac{1}{8} \frac{(k_3^+ - k_2^+)(k_3^+ - q^+)(k_0^+ - k_1^+)}{(q^+ - k_1^+)(k_3^+ - k_1^+)} \log(H_r) \\
& + \frac{1}{4} \frac{k_3^+ - q^+}{(k_3^+ - k_1^+)(q^+ - k_1^+)p_{14}^2} K_{14}^\wedge K_{32}^\vee \log(H_r) \\
& k_3^+ < q^+ < k_1^+ \\
& + \frac{1}{8} \frac{(q^+ - k_2^+)(-q^+ + k_0^+)(k_0^+ - k_1^+)(k_3^+ - k_2^+)}{(k_0^+ - k_2^+)(q^+ - k_1^+)^2} \log(H_s) + \frac{1}{4} \frac{1}{(k_1^+ - k_2^+)p_{14}^2} K_{43}^\wedge K_{32}^\vee \log(H_s)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{8} \frac{(k_3^+ - k_2^+)(k_3^+ - q^+)(k_0^+ - k_1^+)}{(q^+ - k_1^+)(k_3^+ - k_1^+)} \log(H_t) + \frac{1}{4} \frac{1}{(k_1^+ - k_2^+)p_{14}^2} K_{43}^\wedge K_{32}^\vee \log(H_t) \\
& - \frac{1}{8} \frac{(q^+ - k_2^+)(k_3^+ - q^+)(k_0^+ - k_1^+)}{(q^+ - k_1^+)^2} \log(H_d) + \frac{1}{4} \frac{k_3^+ - q^+}{(q^+ - k_1^+)(k_3^+ - k_2^+)p_{14}^2} K_{43}^\wedge K_{32}^\vee \log(H_d) \\
& - \frac{1}{4} \frac{(q^+ - k_2^+)(k_3^+ - k_1^+)}{(k_1^+ - k_2^+)p_{14}^2 (q^+ - k_1^+)(k_3^+ - k_2^+)} K_{43}^\wedge K_{32}^\vee \log(H_l) - \frac{1}{4} \frac{q^+ - k_2^+}{(q^+ - k_1^+)p_{14}^2 (k_1^+ - k_2^+)} K_{14}^\wedge K_{32}^\vee \log(H_u) \\
& - \frac{1}{8} \frac{(k_3^+ - k_2^+)(k_3^+ - q^+)(k_0^+ - k_1^+)}{(q^+ - k_1^+)(k_3^+ - k_1^+)} \log(H_r) + \frac{1}{4} \frac{k_3^+ - q^+}{(k_3^+ - k_1^+)(q^+ - k_1^+)p_{14}^2} K_{14}^\wedge K_{32}^\vee \log(H_r) \\
& k_1^+ < q^+ < k_2^+ \\
& + \frac{1}{8} \frac{(k_3^+ - k_2^+)(-q^+ + k_0^+)(q^+ - k_2^+)(k_0^+ - k_1^+)}{(q^+ - k_1^+)^2 (k_0^+ - k_2^+)} \log(H_s) - \frac{1}{4} \frac{-q^+ + k_0^+}{p_{14}^2 (k_0^+ - k_1^+)(q^+ - k_1^+)} K_{14}^\wedge K_{32}^\vee \log(H_s) \\
& - \frac{1}{4} \frac{1}{(k_0^+ - k_1^+)p_{14}^2} K_{14}^\wedge K_{43}^\vee \log(H_s) - \frac{1}{8} \frac{(k_3^+ - q^+)(q^+ - k_2^+)(k_0^+ - k_1^+)}{(q^+ - k_1^+)^2} \log(H_d) \\
& + \frac{1}{4} \frac{k_3^+ - q^+}{(q^+ - k_1^+)(k_3^+ - k_2^+)p_{14}^2} K_{43}^\wedge K_{32}^\vee \log(H_d) - \frac{1}{4} \frac{(q^+ - k_2^+)(k_3^+ - k_1^+)}{p_{14}^2 (q^+ - k_1^+)(k_1^+ - k_2^+)(k_3^+ - k_2^+)} K_{43}^\wedge K_{32}^\vee \log(H_l) \\
& + \frac{1}{4} \frac{q^+ - k_2^+}{(k_1^+ - k_2^+)(q^+ - k_1^+)p_{14}^2} K_{14}^\wedge K_{32}^\vee \log(H_u)
\end{aligned}$$

Again, we can integrate out whatever can be integrated out, and sweep the rest underneath the infra red term rug. Thus we have:

$$\frac{1}{8\pi^2} g^4 \log \frac{p_{12}^2}{p_{14}^2} \left[ -\frac{1}{8} \frac{(k_2^+ - k_3^+)(k_0^+ - k_3^+)(k_1^+ - k_0^+)}{k_1^+ - k_3^+} - \frac{1}{4} \frac{k_1^+ - k_0^+}{(k_1^+ - k_3^+)p_{14}^2} K_{14}^\wedge K_{32}^\vee \right]$$

The combination of:

$$\begin{aligned}
& \frac{A}{2} \frac{K_{5,2}^\wedge K_{3,2}^\vee p_{14}^2}{q_0^2 q_1^2 q_2^2 p_{14}^2} + \frac{A}{2} \frac{K_{5,2}^\wedge K_{4,3}^\vee p_{14}^2}{q_0^2 q_1^2 q_3^2 p_{12}^2} \\
& + \frac{A}{2} \frac{K_{5,2}^\wedge K_{3,5}^\vee}{(q^+ - k_0^+)^2} \left[ \frac{(-k_0^+ + k_3^+)(-k_0^+ + q^+)}{q_0^2 q_2^2 q_3^2} + \frac{(-k_1^+ + k_0^+)(k_0^+ - q^+)}{q_0^2 q_1^2 q_2^2} \right] \\
& + \frac{A}{2} \frac{K_{5,2}^\wedge K_{3,5}^\vee}{(q^+ - k_0^+)^2} \frac{(k_0^+ - k_3^+)(q^+ - k_1^+) + (k_1^+ - k_0^+)(k_3^+ - q^+)}{q_3^2 q_2^2 q_1^2} - A \frac{K_{5,2}^\wedge K_{3,5}^\vee}{(q^+ - k_0^+)^2} \frac{K_{6,4}^\vee K_{1,6}^\wedge}{q_3^2 q_0^2 q_2^2 q_1^2}
\end{aligned}$$

This is the rotation by 90 degrees clockwise of the previous one. Its contribution is:

$$\begin{aligned}
& \frac{1}{8\pi^2} g^4 \times \\
& k_0^+ < q^+ < k_3^+ \\
& - \frac{1}{8} \frac{(-k_3^+ + k_2^+)(-k_2^+ + q^+)(k_1^+ - k_0^+)}{(-k_0^+ + q^+)(-k_2^+ + k_0^+)} \log(H_s) - \frac{1}{4} \frac{-k_2^+ + q^+}{(-k_0^+ + q^+)p_{12}^2 (-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_s) \\
& + \frac{1}{4} \frac{-k_2^+ + q^+}{(-k_0^+ + q^+)p_{12}^2 (-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_d) + \frac{1}{8} \frac{(-k_2^+ + k_0^+)(-k_3^+ + k_1^+)(-k_1^+ + q^+)}{(-k_0^+ + q^+)(k_1^+ - k_0^+)} \log(H_u) \\
& - \frac{1}{4} \frac{(-k_1^+ + q^+)(-k_2^+ + k_0^+)}{(-k_0^+ + q^+)(k_1^+ - k_0^+)p_{12}^2 (-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_u) \\
& - \frac{1}{8} \frac{(k_0^+ - k_3^+)(-k_1^+ + q^+)(-k_2^+ + k_1^+)}{(k_1^+ - k_0^+)(-k_0^+ + q^+)} \log(H_r)
\end{aligned}$$



$$\begin{aligned}
& + \frac{1}{4} \frac{(-k_2^+ + k_0^+)(-k_1^+ + q^+)}{(-k_0^+ + q^+)(k_1^+ - k_0^+)p_{12}^2(-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_r) \\
& \quad k_3^+ < q^+ < k_1^+ \\
& - \frac{1}{8} \frac{(q^+ - k_2^+)(k_1^+ - k_0^+)(-k_3^+ + k_2^+)}{(-k_0^+ + q^+)(-k_2^+ + k_0^+)} \log(H_s) + \frac{1}{8} \frac{(-k_3^+ + k_2^+)(-k_3^+ + q^+)(-k_1^+ + q^+)(k_1^+ - k_0^+)}{(-k_3^+ + k_1^+)(-k_0^+ + q^+)^2} \log(H_t) \\
& - \frac{1}{4} \frac{q^+ - k_2^+}{(-k_3^+ + k_2^+)p_{12}^2(-k_0^+ + q^+)} K_{32}^\wedge K_{43}^\vee \log(H_d) - \frac{1}{8} \frac{(q^+ - k_2^+)(k_1^+ - k_0^+)(-k_3^+ + q^+)}{(-k_0^+ + q^+)^2} \log(H_l) \\
& + \frac{1}{4} \frac{q^+ - k_2^+}{(-k_3^+ + k_2^+)p_{12}^2(-k_0^+ + q^+)} K_{32}^\wedge K_{43}^\vee \log(H_l) + \frac{1}{8} \frac{(-k_2^+ + k_0^+)(-k_3^+ + k_1^+)(-k_1^+ + q^+)}{(k_1^+ - k_0^+)(-k_0^+ + q^+)} \log(H_u) \\
& - \frac{1}{4} \frac{(-k_2^+ + k_0^+)(-k_1^+ + q^+)}{(-k_3^+ + k_2^+)p_{12}^2(k_1^+ - k_0^+)(-k_0^+ + q^+)} K_{32}^\wedge K_{43}^\vee \log(H_u) - \frac{1}{8} \frac{(k_0^+ - k_3^+)(-k_1^+ + q^+)(-k_2^+ + k_1^+)}{(k_1^+ - k_0^+)(-k_0^+ + q^+)} \log(H_r) \\
& + \frac{1}{4} \frac{(-k_2^+ + k_0^+)(-k_1^+ + q^+)}{(-k_3^+ + k_2^+)p_{12}^2(k_1^+ - k_0^+)(-k_0^+ + q^+)} K_{32}^\wedge K_{43}^\vee \log(H_r) \\
& \quad k_1^+ < q^+ < k_2^+ \\
& + \left[ -\frac{1}{4} \frac{(-k_2^+ + k_1^+)(-k_3^+ + k_2^+)}{-k_2^+ + k_0^+} + \frac{1}{8} \frac{(k_0^+ - k_3^+)(-k_2^+ + k_1^+)(-k_2^+ + q^+)}{(-k_0^+ + q^+)(-k_2^+ + k_0^+)} \right] \log(H_s) \\
& - \frac{1}{4} \frac{k_0^+ k_1^+ - 2k_3^+ k_1^+ - 2k_0^+ q^+ + q^+ k_3^+ + k_3^+ k_0^+ + q^+ k_1^+}{(k_1^+ - k_0^+)(-k_0^+ + q^+)(k_0^+ - k_3^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \log(H_s) \\
& - \frac{1}{4} \frac{(-k_2^+ + k_0^+)(-k_3^+ + q^+)}{(k_0^+ - k_3^+)(-k_0^+ + q^+)(-k_3^+ + k_2^+)p_{12}^2} K_{32}^\wedge K_{43}^\vee \log(H_s) - \frac{1}{4} \frac{-k_2^+ + q^+}{(-k_0^+ + q^+)p_{12}^2(-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_d) \\
& - \frac{1}{8} \frac{(-k_2^+ + q^+)(k_1^+ - k_0^+)(-k_3^+ + q^+)}{(-k_0^+ + q^+)^2} \log(H_l) + \frac{1}{4} \frac{-k_2^+ + q^+}{(-k_0^+ + q^+)p_{12}^2(-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_l) \\
& - \frac{1}{8} \frac{(-k_2^+ + k_0^+)(-k_3^+ + k_1^+)(-k_1^+ + q^+)}{(-k_0^+ + q^+)(k_1^+ - k_0^+)} \log(H_u) \\
& + \frac{1}{4} \frac{(-k_1^+ + q^+)(-k_2^+ + k_0^+)}{(k_1^+ - k_0^+)p_{12}^2(-k_0^+ + q^+)(-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_u)
\end{aligned}$$

Its finite contribution is:

$$\frac{1}{8\pi^2} g^4 \log \frac{p_{12}^2}{p_{14}^2} \left[ -\frac{1}{8} \frac{(k_2^+ - k_3^+)(k_0^+ - k_3^+)(k_1^+ - k_2^+)}{k_0^+ - k_2^+} + \frac{1}{4} \frac{k_0^+ - k_3^+}{(k_0^+ - k_2^+)p_{12}^2} K_{21}^\wedge K_{43}^\vee \right]$$

The combination of:

$$\begin{aligned}
& \frac{A}{2} \frac{K_{1,6}^\vee K_{2,1}^\wedge p_{12}^2}{q_3^2 q_0^2 q_1^2 p_{12}^2} + \frac{A}{2} \frac{K_{1,6}^\vee K_{3,2}^\wedge p_{12}^2}{q_3^2 q_0^2 q_2^2 p_{14}^2} \\
& - \frac{A}{2} \frac{K_{1,6}^\vee K_{5,2}^\wedge}{(q^+ - k_3^+)^2} \left[ \frac{(-k_3^+ + k_2^+)(-k_3^+ + q^+)}{q_3^2 q_1^2 q_2^2} + \frac{(-k_0^+ + k_3^+)(k_3^+ - q^+)}{q_3^2 q_0^2 q_1^2} \right] \\
& - \frac{A}{2} \frac{K_{1,6}^\vee K_{5,2}^\wedge}{(q^+ - k_3^+)^2} \frac{(k_3^+ - k_2^+)(q^+ - k_0^+) + (k_0^+ - k_3^+)(k_2^+ - q^+)}{q_2^2 q_1^2 q_0^2} - A \frac{K_{1,6}^\vee K_{5,2}^\wedge}{(q^+ - k_3^+)^2} \frac{K_{3,5}^\wedge K_{6,4}^\vee}{q_2^2 q_3^2 q_1^2 q_0^2}
\end{aligned}$$

gives:

$$A \frac{1}{8\pi^2} g^4 \times$$

$$\begin{aligned}
& k_0^+ < q^+ < k_3^+ \\
& - \frac{1}{8} \frac{(-k_2^+ + q^+)(k_1^+ - k_0^+)(-k_3^+ + k_2^+)(-k_0^+ + q^+)}{(-k_2^+ + k_0^+)(-k_3^+ + q^+)^2} \log(H_s) \\
& - \frac{1}{4} \frac{-k_2^+ + q^+}{(-k_3^+ + q^+)(-k_3^+ + k_2^+)p_{14}^2} K_{32}^\wedge K_{14}^\vee \log(H_s) \\
& + \frac{1}{4} \frac{1}{p_{14}^2(-k_3^+ + k_2^+)} K_{32}^\wedge K_{21}^\vee \log(H_s) + \frac{1}{8} \frac{(-k_2^+ + k_1^+)(q^+ - k_0^+)}{-k_3^+ + q^+} \log(H_d) \\
& + \frac{1}{4} \frac{(-k_3^+ + k_1^+)(q^+ - k_0^+)}{(k_0^+ - k_3^+)p_{14}^2(-k_3^+ + k_2^+)(-k_3^+ + q^+)} K_{32}^\wedge K_{43}^\vee \log(H_d) - \frac{1}{8} \frac{(-k_1^+ + q^+)(-k_3^+ + k_2^+)(q^+ - k_0^+)}{(-k_3^+ + q^+)^2} \log(H_u) \\
& + \frac{1}{4} \frac{-k_1^+ + q^+}{p_{14}^2(-k_3^+ + k_2^+)(-k_3^+ + q^+)} K_{32}^\wedge K_{43}^\vee \log(H_u) - \frac{1}{4} \frac{(-k_0^+ + q^+)(-k_3^+ + k_1^+)}{p_{14}^2(-k_3^+ + q^+)(k_0^+ - k_3^+)(-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_r) \\
\\
& k_3^+ < q^+ < k_1^+ \\
& - \frac{1}{8} \frac{(-k_3^+ + k_2^+)(q^+ - k_2^+)(-k_0^+ + q^+)(k_1^+ - k_0^+)}{(-k_2^+ + k_0^+)(-k_3^+ + q^+)^2} \log(H_s) + \frac{1}{4} \frac{k_1^+ - k_0^+}{(-k_3^+ + k_2^+)p_{14}^2(k_0^+ - k_3^+)} K_{32}^\wedge K_{43}^\vee \log(H_s) \\
& + \frac{1}{8} \frac{(k_1^+ - k_0^+)(-k_3^+ + k_2^+)(-k_1^+ + q^+)}{(-k_3^+ + k_1^+)(-k_3^+ + q^+)} \log(H_t) + \frac{1}{4} \frac{k_1^+ - k_0^+}{(-k_3^+ + k_2^+)p_{14}^2(k_0^+ - k_3^+)} K_{32}^\wedge K_{43}^\vee \log(H_t) \\
& - \frac{1}{8} \frac{(-k_2^+ + k_1^+)(-k_0^+ + q^+)}{-k_3^+ + q^+} \log(H_d) - \frac{1}{4} \frac{(-k_0^+ + q^+)(-k_3^+ + k_1^+)}{(-k_3^+ + q^+)p_{14}^2(k_0^+ - k_3^+)(-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_d) \\
& + \frac{1}{8} \frac{(-k_2^+ + k_0^+)(-k_1^+ + q^+)}{-k_3^+ + q^+} \log(H_l) + \frac{1}{4} \frac{-k_1^+ + q^+}{p_{14}^2(-k_3^+ + k_2^+)(-k_3^+ + q^+)} K_{32}^\wedge K_{43}^\vee \log(H_l) \\
& - \frac{1}{8} \frac{(-k_1^+ + q^+)(-k_3^+ + k_2^+)(-k_0^+ + q^+)}{(-k_3^+ + q^+)^2} \log(H_u) + \frac{1}{4} \frac{-k_1^+ + q^+}{p_{14}^2(-k_3^+ + k_2^+)(-k_3^+ + q^+)} K_{32}^\wedge K_{43}^\vee \log(H_u) \\
& - \frac{1}{4} \frac{(-k_0^+ + q^+)(-k_3^+ + k_1^+)}{(-k_3^+ + q^+)p_{14}^2(k_0^+ - k_3^+)(-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_r) \\
\\
& k_1^+ < q^+ < k_2^+ \\
& - \frac{1}{8} \frac{(-k_3^+ + k_2^+)(-k_2^+ + k_1^+)(-k_0^+ + q^+)^2}{(-k_3^+ + q^+)^2(-k_2^+ + k_0^+)} \log(H_s) \\
& + \frac{1}{4} \frac{(-k_0^+ + q^+)(-k_3^+ + k_1^+)}{(k_0^+ - k_3^+)p_{14}^2(-k_3^+ + q^+)(-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_s) \\
& - \frac{1}{8} \frac{(-k_2^+ + k_1^+)(-k_0^+ + q^+)}{-k_3^+ + q^+} \log(H_d) - \frac{1}{4} \frac{(-k_0^+ + q^+)(-k_3^+ + k_1^+)}{(k_0^+ - k_3^+)p_{14}^2(-k_3^+ + q^+)(-k_3^+ + k_2^+)} K_{32}^\wedge K_{43}^\vee \log(H_d) \\
& + \frac{1}{8} \frac{(-k_2^+ + k_0^+)(-k_1^+ + q^+)}{-k_3^+ + q^+} \log(H_l) + \frac{1}{4} \frac{-k_1^+ + q^+}{(-k_3^+ + q^+)(-k_3^+ + k_2^+)p_{14}^2} K_{32}^\wedge K_{43}^\vee \log(H_l) \\
& + \frac{1}{8} \frac{(-k_1^+ + q^+)(-k_3^+ + k_2^+)(-k_0^+ + q^+)}{(-k_3^+ + q^+)^2} \log(H_u) - \frac{1}{4} \frac{-k_1^+ + q^+}{(-k_3^+ + q^+)(-k_3^+ + k_2^+)p_{14}^2} K_{32}^\wedge K_{43}^\vee \log(H_u)
\end{aligned}$$

Its finite contribution is:

$$\frac{1}{8\pi^2} g^4 \log \frac{p_{12}^2}{p_{14}^2} \left[ -\frac{1}{8} \frac{(k_2^+ - k_3^+)(k_1^+ - k_2^+)(k_1^+ - k_0^+)}{k_1^+ - k_3^+} - \frac{1}{4} \frac{k_2^+ - k_3^+}{(k_1^+ - k_3^+)p_{14}^2} K_{32}^\wedge K_{14}^\vee \right]$$

The combination of:

$$\begin{aligned}
& -\frac{A}{2} \frac{K_{6,4}^\wedge K_{1,4}^\vee p_{14}^2}{q_2^2 q_3^2 q_0^2 p_{14}^2} - \frac{A}{2} \frac{K_{6,4}^\wedge K_{2,1}^\vee p_{14}^2}{q_2^2 q_3^2 q_1^2 p_{12}^2} \\
& + \frac{A}{2} \frac{K_{6,4}^\wedge K_{1,6}^\vee}{(q^+ - k_2^+)^2} \left[ \frac{(-k_2^+ + k_1^+)(-k_2^+ + q^+)}{q_2^2 q_0^2 q_1^2} + \frac{(-k_3^+ + k_2^+)(k_2^+ - q^+)}{q_2^2 q_3^2 q_0^2} \right] \\
& + \frac{A}{2} \frac{K_{6,4}^\wedge K_{1,6}^\vee}{(q^+ - k_2^+)^2} \frac{(k_2^+ + k_1^+)(q^+ - k_3^+) + (k_3^+ - k_2^+)(q^+ - k_3^+)}{q_1^2 q_0^2 q_3^2} - A \frac{K_{6,4}^\wedge K_{1,6}^\vee}{(q^+ - k_2^+)^2} \frac{K_{5,2}^\vee K_{3,5}^\wedge}{q_1^2 q_2^2 q_0^2 q_1^2}
\end{aligned}$$

gives:

$$\begin{aligned}
& A \frac{1}{8\pi^2} g^4 \times \\
& k_0^+ < q^+ < k_3^+ \\
& + \left[ -\frac{1}{4} \frac{(k_0^+ - k_3^+)(k_1^+ - k_0^+)}{k_0^+ - k_2^+} + \frac{1}{8} \frac{(k_1^+ - k_2^+)(k_0^+ - k_3^+)(k_0^+ - q^+)}{(k_2^+ - q^+)(k_0^+ - k_2^+)} \right] \log(H_s) \\
& + \frac{1}{4} \frac{1}{(k_1^+ - k_2^+) p_{12}^2} K_{14}^\wedge K_{21}^\vee \log(H_s) - \frac{1}{4} \frac{k_1^+ k_2^+ - 2k_1^+ k_3^+ - 2q^+ k_2^+ + q^+ k_3^+ + q^+ k_1^+ + k_2^+ k_3^+}{(k_1^+ - k_2^+)(k_2^+ - q^+)(k_2^+ - k_3^+) p_{12}^2} K_{43}^\wedge K_{21}^\vee \log(H_s) \\
& - \frac{1}{4} \frac{k_0^+ - q^+}{(k_2^+ - q^+)(k_2^+ - k_3^+) p_{12}^2} K_{43}^\wedge K_{32}^\vee \log(H_s) - \frac{1}{8} \frac{(k_0^+ - k_2^+)(k_1^+ - k_3^+)(-q^+ + k_3^+)}{(k_2^+ - k_3^+)(k_2^+ - q^+)} \log(H_d) \\
& + \frac{1}{4} \frac{(k_0^+ - k_2^+)(-q^+ + k_3^+)}{(k_2^+ - q^+) p_{12}^2 (k_2^+ - k_3^+)^2} K_{43}^\wedge K_{32}^\vee \log(H_d) - \frac{1}{8} \frac{(k_0^+ - k_3^+)(k_1^+ - k_2^+)(k_0^+ - q^+)}{(k_2^+ - q^+)(k_0^+ - k_2^+)} \log(H_u) \\
& + \frac{1}{4} \frac{k_0^+ - q^+}{(k_0^+ - k_2^+)(k_2^+ - q^+) p_{12}^2} K_{43}^\wedge K_{21}^\vee \log(H_u) - \frac{1}{8} \frac{(k_2^+ - k_3^+)(k_0^+ - q^+)(k_1^+ - q^+)}{(k_2^+ - q^+)^2} \log(H_r) \\
& + \frac{1}{4} \frac{k_0^+ - q^+}{(k_1^+ - k_0^+)(k_2^+ - q^+) p_{12}^2} K_{14}^\wedge K_{21}^\vee \log(H_r) \\
& k_3^+ < q^+ < k_1^+ \\
& - \frac{1}{8} \frac{(k_0^+ - q^+)(k_2^+ - k_3^+)(k_1^+ - k_0^+)}{(k_2^+ - q^+)(k_0^+ - k_2^+)} \log(H_s) + \frac{1}{8} \frac{(k_3^+ - q^+)(k_2^+ - k_3^+)(-q^+ + k_1^+)(k_1^+ - k_0^+)}{(k_1^+ - k_3^+)(k_2^+ - q^+)^2} \log(H_t) \\
& + \frac{1}{8} \frac{(k_0^+ - k_2^+)(k_1^+ - k_3^+)(k_3^+ - q^+)}{(k_2^+ - k_3^+)(k_2^+ - q^+)} \log(H_d) - \frac{1}{4} \frac{(k_3^+ - q^+)(k_0^+ - k_2^+)}{p_{12}^2 (k_2^+ - k_3^+)^2 (k_2^+ - q^+)} K_{43}^\wedge K_{32}^\vee \log(H_d) \\
& - \frac{1}{4} \frac{k_3^+ - q^+}{p_{12}^2 (k_2^+ - q^+)(k_2^+ - k_3^+)} K_{43}^\wedge K_{21}^\vee \log(H_l) - \frac{1}{8} \frac{(k_0^+ - q^+)(k_0^+ - k_3^+)(k_1^+ - k_2^+)}{(k_2^+ - q^+)(k_0^+ - k_2^+)} \log(H_u) \\
& + \frac{1}{4} \frac{k_0^+ - q^+}{(k_0^+ - k_2^+)(k_2^+ - q^+) p_{12}^2} K_{43}^\wedge K_{21}^\vee \log(H_u) - \frac{1}{8} \frac{(k_2^+ - k_3^+)(k_0^+ - q^+)(-q^+ + k_1^+)}{(k_2^+ - q^+)^2} \log(H_r) \\
& + \frac{1}{4} \frac{k_0^+ - q^+}{(k_1^+ - k_0^+)(k_2^+ - q^+) p_{12}^2} K_{14}^\wedge K_{21}^\vee \log(H_r) \\
& k_1^+ < q^+ < k_2^+ \\
& - \frac{1}{8} \frac{(k_1^+ - k_0^+)(k_2^+ - k_3^+)(k_0^+ - q^+)}{(k_0^+ - k_2^+)(k_2^+ - q^+)} \log(H_s) - \frac{1}{4} \frac{k_0^+ - q^+}{(k_2^+ - q^+)(k_2^+ - k_3^+) p_{12}^2} K_{43}^\wedge K_{32}^\vee \log(H_s) \\
& + \frac{1}{8} \frac{(k_0^+ - k_2^+)(k_1^+ - k_3^+)(k_3^+ - q^+)}{(k_2^+ - k_3^+)(k_2^+ - q^+)} \log(H_d) - \frac{1}{4} \frac{(k_0^+ - k_2^+)(k_3^+ - q^+)}{(k_2^+ - q^+)(k_2^+ - k_3^+)^2 p_{12}^2} K_{43}^\wedge K_{32}^\vee \log(H_d) \\
& - \frac{1}{4} \frac{k_3^+ - q^+}{(k_2^+ - q^+)(k_2^+ - k_3^+) p_{12}^2} K_{43}^\wedge K_{21}^\vee \log(H_l) + \frac{1}{8} \frac{(k_0^+ - q^+)(k_0^+ - k_3^+)(k_1^+ - k_2^+)}{(k_2^+ - q^+)(k_0^+ - k_2^+)} \log(H_u)
\end{aligned}$$

$$- \frac{1}{4} \frac{k_0^+ - q^+}{(k_0^+ - k_2^+)(k_2^+ - q^+)p_{12}^2} K_{43}^\wedge K_{21}^\vee \log(H_u)$$

Its finite contribution is:

$$\frac{1}{8\pi^2} g^4 \log \frac{p_{12}^2}{p_{14}^2} \left[ -\frac{1}{8} \frac{(k_0^+ - k_3^+)(k_1^+ - k_0^+)(k_1^+ - k_2^+)}{k_0^+ - k_2^+} + \frac{1}{4} \frac{k_1^+ - k_2^+}{(k_0^+ - k_2^+)p_{12}^2} K_{43}^\wedge K_{21}^\vee \right]$$

## References

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